A simpler proof for the two disjoint odd cycles theorem

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ABSTRACT

We give a short proof of the two disjoint odd cycles theorem which characterizes graphs without two vertex-disjoint odd cycles. Our proof does not depend on any matroid result. It only uses the two paths theorem, which characterizes graphs without two disjoint paths with specified ends (i.e, 2-linked graphs).

Keywords: Two disjoint odd cycles and projective plane.

1 The two disjoint odd cycles theorem

A characterization of graphs without an odd cycle is easy, as it is exactly bipartite. However, graphs without two vertex-disjoint odd cycles are not so simple. Indeed, one of the graphs can be, roughly, embedded into the projective plane. This graph attracts a lot of attention by many researchers in graph theory and combinatorial optimization, because it appears in many contexts. Let us give a few examples:

(1) It has no two vertex-disjoint odd cycles. However, it needs at least $\sqrt{n}$ vertices to hit all odd cycles. Hence it shows that the well-known Erdős-Pósa property does not hold for odd cycles.

(2) It contains an $O(\sqrt{n})$ half-integral odd cycles packing. Thus this shows large integrality gap ($O(\sqrt{n})$) for the odd cycles packing problem (roughly the ratio between fractional packing and integral packing).

(3) It can be easily modified to give an example which shows large integrality gap ($O(\sqrt{n})$) for the well-known (maximum) disjoint paths problem (even for planar graphs).

(4) If all faces are 4-cycle, then this graph appears many places in topological graph theory (esp. graph coloring). For more details, we refer the reader to the book [2].

Therefore, the characterization of graphs without two vertex-disjoint odd cycles is well-known. On the other hand, its proof is less known. Indeed, Lovász (see [5]) is the first to give a complete proof for this characterization, however, his proof heavily depends on the seminal result by Seymour [3] for

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decomposing regular matroids. In fact, his proof is not published, yet Gerards, Lovász, Schrijver, Seymour and Truemper were trying to write up a proof around 1990.

In this paper, we give a new, simpler proof which only depends on the two paths theorem [4, 6, 7], which characterizes graphs without two vertex-disjoint paths with specified ends (i.e, 2-linked graphs). In addition, our proof is simpler and shorter. Furthermore, it can also extend to a characterization of signed graphs without two vertex-disjoint negative cycles.

We learned that Seymour (private communication) also has a different proof, which does not depend on any matroid results, but depends on the characterization of graphs without odd K_4-minor by Gerards [1].

Let us now mention the characterization of graphs without two vertex-disjoint odd cycles. To do so, we need some definitions.

A *separation* in a graph G is a pair (K_1, K_2) of subgraphs of G such that $G = K_1 \cup K_2$, $E(K_1) \cap E(K_2) = \emptyset$, and $E(K_i) \cup V(K_i - K_{3-i}) \neq \emptyset$ for $i = 1, 2$. If, in addition, $|K_1 \cap K_2| = k$, then (K_1, K_2) is a *k-separation* in G. A graph G is called *internally 4-connected* if G is 3-connected and for every 3-separation (K_1, K_2) in G, $|K_1| \leq 4$ or $|K_2| \leq 4$.

We are now ready to mention the characterization of graphs without two vertex-disjoint odd cycles.

Theorem 1 *Let G be an internally 4-connected graph. Then G has no two vertex-disjoint odd cycles if and only if G satisfies one of the following;*

(i) $G - \{x\}$ *is bipartite for some vertex* $x \in V(G)$,

(ii) $G - \{e_1, e_2, e_3\}$ *is bipartite for some edges* $e_1, e_2, e_3 \in E(G)$ *such that* e_1, e_2, e_3 *form a triangle,*

(iii) $|G| \leq 5$, *and*

(iv) *G can be embedded into the projective plane so that every face boundary has even length.*

We need some notations. Let G be a graph. Two edges are called *independent* in G if they have no common end vertex. For a vertex u in G, we denote the set of neighbors of u by $N_G(u)$, and for $A \subset V(G)$, let $N_G(A) = \bigcup_{u \in A} N_G(u) - A$.

For a plane graph G, we denote the outer face boundary cycle by ∂G. Moreover, for $u, v \in \partial G$, $u\partial Gv$ is the subpath of ∂G connecting u and v in the clockwise order.

To this end, we give one easy lemma, which shows the existence of a spanning 2-connected bipartite subgraph in a 3-connected graph. This lemma serves as a basis of our approach.

Lemma 2 *Let G be a 3-connected graph. Then there exists a 2-connected spanning bipartite subgraph of G.*

Proof. Take any two vertices of G and consider three internally vertex-disjoint paths connecting them. Since at least two of the three paths have length of the same parity, we can find an even cycle in G, that is, a 2-connected bipartite subgraph of G.

We take a 2-connected bipartite subgraph H of G so that $|H|$ is as large as possible. Suppose that there exists a vertex $u \in V(G) - V(H)$.

Since G is 3-connected, we can find three paths P_1, P_2, P_3 from u to $V(H)$ with $V(P_i) \cap V(P_j) = \{u\}$ for any $1 \leq i < j \leq 3$. It is again straightforward to check that either $H \cup P_1 \cup P_2$ or $H \cup P_2 \cup P_3$ or $H \cup P_3 \cup P_1$ is a bipartite graph, which contradicts the maximality of H. $\square$

As mentioned above, our proof requires the characterization of graphs without two vertex-disjoint paths with specified endvertices. Thus we shall give this characterization in the next section.

2 The two paths theorem

We now give a characterization of the two paths theorem. Let H be a graph and let $A_1, \ldots, A_l \subseteq V(H)$ be pairwise vertex-disjoint sets and let $\mathbf{A} = \{A_1, \ldots, A_l\}$. We call $\mathbf{A}$ a 3-*separated set* of H if

(1) for $1 \le i, j \le l$ with $i \ne j$, $N_H(A_i) \cap A_j = \emptyset$, and

(2) for $1 \le i \le l$, $|N_H(A_i)| \le 3$.

We say that H *can be embedded into the plane, with respect to* $\mathbf{A}$, if $H(\mathbf{A})$ may be drawn in the plane, where $H(\mathbf{A})$ is the graph obtained from H by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N_H(A_i)$.

We can now mention the two path theorem [4, 6, 7].

Theorem 3 *Let H be a graph, and suppose four vertices s_1, t_1, s_2, t_2 are given. Either*

(1) *there are two vertex-disjoint paths P_1, P_2 such that P_i connects s_i, t_i for $i = 1, 2$, or*

(2) *there is a 3-separated set $\mathbf{A}$ of H with $s_1, t_1, s_2, t_2 \notin A$ for any $A \in \mathbf{A}$, and there is an embedding of H in the plane, with respect to $\mathbf{A}$, such that $\partial H(\mathbf{A})$ contains the four vertices s_1, s_2, t_1, t_2 in clockwise order.*

We can indeed choose $\mathbf{A}$ in Theorem 3 (2) so that, the following property holds:

If $|N_H(A_i)| = 3$, then $N_H(A_i)$ induces a facial triangle in $H(\mathbf{A})$.

To see this, we may choose $\mathbf{A}$ such that, the number of non-facial triangles in $H(\mathbf{A})$ induced by members of $\mathbf{A}$ is minimum. Suppose, without loss of generality, that $|N_H(A_1)| = 3$ and $N_H(A_1)$ induces a triangle T_1 in $H(\mathbf{A})$, which is not facial. Let $D_1 \subseteq V(H(\mathbf{A}))$ be such that, for each $x \in V(H)$, $x \in D_1$ if and only if x is contained in the closed disk bounded by T_1. Define $A_1' \subseteq V(H)$ such that, for each $x \in V(H)$, $x \in A_1'$ if and only if $x \in D_1 - N_H(A_1)$ or $x \in A_j$ for some A_j with $N_H(A_j) \subseteq D_1$. Let $\mathbf{A}' = (\mathbf{A} - \{A_j : N_H(A_j) \subseteq D_1\}) \cup \{A_1'\}$. Then $\mathbf{A}'$ is a 3-separated set, but the number of non-facial triangles in $H(\mathbf{A}')$ is smaller than that of $H(\mathbf{A})$, a contradiction.

When Theorem 3 (2) holds, we take a 3-separated set $\mathbf{A}$ so that $\mathbf{A}$ is as small as possible, which we refer to as "*minimal*". This means that no 3-separated set $\mathbf{A}'$ exists satisfying all of the following: $\mathbf{A}' \ne \mathbf{A}$, for every $A' \in \mathbf{A}'$ there exists $A \in \mathbf{A}$ such that $A' \subset A$, and $H(\mathbf{A}')$ exhibits the same desired conditions as $H(\mathbf{A})$. Such $\mathbf{A}$ and $H(\mathbf{A})$ have certain linkage property. The following is equivalent to Proposition 3.2 in [8].

Proposition 4 (Yu [8]) *Let H be a graph, and let $\mathbf{A}$ be a minimal 3-separated set of H so that $H(\mathbf{A})$ is a plane graph. Let $s_1, s_2 \in V\big(H(\mathbf{A})\big)$ and $t_1^*, t_2^* \in V(H)$. Let t_i be a vertex in $A_i \cup N_H(A_i)$ if $t_i^* \in N_H(A_i)$ for some $A_i \in \mathbf{A}$ with $A_1 \ne A_2$; otherwise $t_i = t_i^*$. Suppose that there exist two vertex-disjoint paths P_1^*, P_2^* in $H(\mathbf{A})$ such that P_i^* connects s_i and t_i^*. Then H has two vertex-disjoint paths P_1 and P_2 such that P_i connects s_i and t_i, and $V\big(P_i \cap H(\mathbf{A})\big) = V(P_i^*)$.*

We now give another two propositions that are needed in our proof.

Proposition 5 *Let H be a graph, and let $\mathbf{A}$ be a minimal 3-separated set of H so that $H(\mathbf{A})$ is a plane graph. Suppose there is a separation (K_1, K_2) of order two or three in H with $|K_1| \le 4$ and $K_1 \cap K_2 \subset V(H(\mathbf{A}))$. Then $K_1 \cap A = \emptyset$ for any $A \in \mathbf{A}$. In other words, K_1 is contained in the plane graph $H(\mathbf{A})$.*

Proof of Proposition 5. Suppose that $K_1 \cap A \neq \emptyset$ for some $A \in \mathbf{A}$. Let $A' = K_1 \cap A$. By our assumption that $K_1 \cap K_2 \subset V(H(\mathbf{A}))$, $|A'| \leq |K_1 - K_2| \leq 2$. By adding A' to the face that contains $K_1 \cap K_2 \subset V(H(\mathbf{A}))$, we can obtain a new planar embedding of H, with respect to $(\mathbf{A} - \{A\}) \cup \{A - A'\}$. This contradicts the minimality of $\mathbf{A}$. □

Proposition 6 *Suppose H is a 2-connected plane graph, and suppose four vertices s_1, t_1, s_2, t_2 are given. Suppose furthermore that s_1 is not contained in ∂H. If there are no two vertex-disjoint paths P_1, P_2 such that P_i connects s_i, t_i for $i = 1, 2$, then there is a 2-separation (K_1, K_2) such that $K_1 \cap K_2 \subset V(\partial H)$, s_1 is in $K_1 - K_2$ and K_2 contains all of s_1, t_1, t_2.*

The proof of this proposition is straightforward by the 2-connectivity and Theorem 3 (2).

We are now ready to give our proof of Theorem 1.

3 Proof of the Main Theorem

Proof of Theorem 1. The "if" part can be easily checked. Hence we shall only show the "only if" part. Let G be an internally 4-connected graph having no two vertex-disjoint odd cycles. If $|G| \leq 8$, then we can easily check that one of (i)–(iv) holds, and hence we may assume the following;

$$|G| \geq 9. \tag{1}$$

We take a 2-connected spanning bipartite subgraph H of G so that $|E(H)|$ is as large as possible. By Lemma 2, such a spanning subgraph H exists.

Note that for any edge uv in $E(G) - E(H)$, both u and v are contained in the same partite set of H, since otherwise we can add the edge uv into H, which contradicts the choice of H. This implies the following fact, which will be often used in the proof.

Fact 1 *For any edge $uv \in E(G) - E(H)$, and for any path P of H connecting u and v, $uPvu$ is an odd cycle.*

Suppose that there exist two independent edges u_1v_1 and u_2v_2 in $E(G) - E(H)$. If there exist two vertex-disjoint paths P_1 and P_2 in H such that P_i connects u_i and v_i for $i = 1, 2$, then $u_1P_1v_1u_1$ and $u_2P_2v_2u_2$ are two vertex-disjoint odd cycles, a contradiction. Thus, there are no such vertex-disjoint paths. Hence we obtain the following claim.

Claim 2 *For any two independent edges u_1v_1 and u_2v_2 in $E(G) - E(H)$, there exist no two vertex-disjoint paths P_1 and P_2 in H such that P_i connects u_i and v_i for $i = 1, 2$,*

If there are no two independent edges in $E(G) - E(H)$, then (i) or (ii) of Theorem 1 holds. Thus, there are two independent edges u_1v_1 and u_2v_2 in $E(G) - E(H)$. By Claim 2 and by Theorem 3, there is a 3-separated set $\mathbf{A}$ of H with $u_1, v_1, u_2, v_2 \notin A$ for any $A \in \mathbf{A}$, and there is an embedding of H in a plane, with respect to $\mathbf{A}$, such that $\partial H(\mathbf{A})$ contains the four vertices u_1, u_2, v_1, v_2 in clockwise order. Let

$$S = \{u \in V(H) : u \text{ is an end vertex of an edge in } E(G) - E(H)\}.$$

We take such a 3-separated set $\mathbf{A}$ and an embedding of H into the plane, with respect to $\mathbf{A}$, so that

(A1) $\mathbf{A}$ is minimal,

(A2) $|\partial H(\mathbf{A}) \cap S|$ is as large as possible, subject to (A1).

At a high level, we are going to prove that all the missing edges can be placed in the boundary $\partial H(\mathbf{A})$ so that they form the crosscap. Then this graph would be one of the graphs described in Theorem 1.

Since H is 2-connected, clearly $H(\mathbf{A})$ is 2-connected. By the choice (A1), Propositions 4 and 5 can be applied.

Our main claim in the proof is the following;

$$\text{either } G \text{ satisfies (ii) of Theorem 1 or } S \subset \partial H(\mathbf{A}). \tag{2}$$

Indeed, once we show (2), we can complete our proof of the main theorem. To this end, assume (2) holds and G does not satisfy (ii) of Theorem 1. We first show that $\mathbf{A} = \emptyset$. Suppose for a contradiction that $\mathbf{A} \neq \emptyset$, say $A \in \mathbf{A}$. By (2), $S \cap A = \emptyset$, and hence $N_G(A) = N_H(A)$. Then $(\widetilde{A}, V(G) - A)$ is a 3-separation in G with $|V(G) - A| \geq 4$, where $\widetilde{A} = A \cup N_G(A)$. If $|V(G) - A| \geq 5$, then $|\widetilde{A}| \leq 4$ since G is internally 4-connected. But this contradicts Proposition 5 since $(\widetilde{A}, V(G) - A)$ is also a 2- or 3-separation in H. So $|V(G) - A| = 4$, and hence $V(G) - A = \{u_1, u_2, v_1, v_2\}$. Since $|N_G(A)| = |N_H(A)| \leq 3$, we may assume that $v_2 \notin N_G(A)$. It follows from the 3-connectedness of G that $N_G(v_2) = \{u_1, u_2, v_1\}$. Note that $G - \{u_1v_1, u_1v_2, v_1v_2\} = (H - \{u_1v_2, v_1v_2\}) \cup \{u_2v_2\}$. Then u_1v_1, u_1v_2, v_1v_2 form a triangle such that $G - \{u_1v_1, u_1v_2, v_1v_2\}$ is bipartite, and hence (ii) of Theorem 1 holds. Thus we have

$$\mathbf{A} = \emptyset. \tag{3}$$

Now we add a crosscap into the outer face of $H(\mathbf{A}) = H$, and are going to embed all edges in $E(G) - E(H)$ in this crosscap. Note that any face boundary of H has even length by our construction. Therefore if we can embed all edges in $E(G) - E(H)$ in the crosscap, G satisfies the condition (iv) and we are done.

Let $S = \{u_1, u_2, \ldots, u_l\}$ such that $u_1, u_2, \ldots, u_l$ appear in ∂H in the clockwise order. If there are two independent edges $u_1u_j, u_pu_q \in E(G) - E(H)$ with $1 < j < p < q \leq l$, then we can easily find two vertex-disjoint paths in H connecting u_1 and u_j, and u_p and u_q, along ∂H, a contradiction to Claim 2. Therefore, no two independent edges $u_iu_j, u_pu_q \in E(G) - E(H)$ with u_i, u_j, u_p, u_q appearing in $\partial H(\mathbf{A})$ in this order, exist. If there are two edges $u_1u_i, u_iu_j \in E(G) - E(H)$ with $1 < i < j$, then for any u_p with $j \leq p \leq l$ and for any u_q with $q \neq i$, we have $u_pu_q \notin E(G) - E(H)$; since otherwise u_p, u_q, u_i, u_j or u_1, u_i, u_q, u_p appear in ∂H in this order, contradicting the above facts.

These facts imply that all vertices of S appear in a "desired" order in ∂H, and hence we can embed all edges of $E(G) - E(H)$ in the crosscap. This completes the proof of Theorem 1.

It remains to show (2). Suppose that $S - \partial H(\mathbf{A}) \neq \emptyset$, say $u \in S - \partial H(\mathbf{A})$, and let uv be the edge in $E(G) - E(H)$. Since $u \neq u_1, u_2, v_1, v_2$, at least one of u_1v_1 and u_2v_2 is independent with uv, say u_1v_1.

Claim 3 *There is a 2-separation (K_1, K_2) in H such that $(K_1 - K_2) \cap \partial H(\mathbf{A}) \neq \emptyset$, $(K_2 - K_1) \cap \partial H(\mathbf{A}) \neq \emptyset$, $u \in K_1 - K_2 - \partial H(\mathbf{A})$, and $u_1, v_1, v \in K_2$.*

Proof. Suppose for a contradiction that such a separation does not exist. Let $u^* = u$ if $u \notin A$ for any $A \in \mathbf{A}$; otherwise let u^* be any vertex in $N_H(A)$ with $u \in A \in \mathbf{A}$. Similarly, we define $v^* \in V(H(\mathbf{A}))$ for v.

By our assumption that Claim 3 does not hold, $H(\mathbf{A})$ has no 2-separation (K_1^*, K_2^*) such that $(K_1^* - K_2^*) \cap \partial H(\mathbf{A}) \neq \emptyset$, $(K_2^* - K_1^*) \cap \partial H(\mathbf{A}) \neq \emptyset$, $u^* \in K_1^* - K_2^* - \partial H(\mathbf{A})$, and $u_1, v_1, v^* \in K_2^*$, either. This implies that there is a path P^* in $H(\mathbf{A})$ connecting u^* and v^* such that either $V(P^*) \cap u_1\partial H(\mathbf{A})v_1 = \emptyset$, or $V(P^*) \cap v_1\partial H(\mathbf{A})u_1 = \emptyset$. We may assume that $V(P^*) \cap u_1\partial H(\mathbf{A})v_1 = \emptyset$, and let $Q^* = u_1\partial H(\mathbf{A})v_1$. By Proposition 4, there are two vertex-disjoint paths P and Q in H connecting u and v, and u_1 and v_1, that are extended from P^*, Q^*, respectively. This contradicts Claim 2. □

Let (K_1, K_2) be a 2-separation as in Claim 3, and let $\{x, y\} = K_1 \cap K_2$. By the assumption of Claim 3, $x, y \in \partial H(\mathbf{A})$. We next claim the following.

Claim 4 $\partial H(\mathbf{A}) \cap (K_1 - K_2) \cap S \neq \emptyset$.

Proof. Suppose that $\partial H(\mathbf{A}) \cap (K_1 - K_2) \cap S = \emptyset$. Note that $u \in K_1 - K_2 - \partial H(\mathbf{A})$. Let $\mathbf{A}' = \{A \in \mathbf{A} : A \cap K_2 \neq \emptyset\} \cup \{K_1 - \{x, y, u\}\}$. Then we obtain that $\mathbf{A}'$ is a 3-separated set of H with $u_1, v_1, u_2, v_2 \notin A$ for any $A \in \mathbf{A}$, and there is an embedding of H in a plane, with respect to $\mathbf{A}'$, satisfying the conditions as in $\mathbf{A}$. Then it is clear that there exists a collection $\mathbf{B}$ such that $B \subset K_1 - \{x, y, u\}$ for all $B \in \mathbf{B}$ and $\mathbf{A}'' = \{A \in \mathbf{A} : A \cap K_2 \neq \emptyset\} \cup \mathbf{B}$ is minimal. Since u appears in $\partial H(\mathbf{A}'')$, $|\partial H(\mathbf{A}'') \cap S| \geq |\partial H(\mathbf{A}) \cap S| + 1$, which contradicts (A2). □

We take such a 2-separation (K_1, K_2) in H so that $|K_1|$ is as small as possible. By Claim 4, there is a vertex $u' \in \partial H(\mathbf{A}) \cap (K_1 - K_2) \cap S$, and let v' be a vertex with $u'v' \in E(G) - E(H)$. Let P'_{ux} and P'_{uy} be the paths in K_1 corresponding to $u'\partial H(\mathbf{A})x$ and $y\partial H(\mathbf{A})u'$, respectively.

Since $\{x, y\} = K_1 \cap K_2$, we may assume that $K_1 \cap \partial H(\mathbf{A}) = y\partial H(\mathbf{A})x$ and $K_2 \cap \partial H(\mathbf{A}) = x\partial H(\mathbf{A})y$. Since we took $|K_1|$ to be minimal, there exist paths P_{ux} and P_{uy} in $K_1 - (\partial H(\mathbf{A}) - \{x, y\})$ that connect u with x and y, respectively. Clearly P_{ux}, P_{uy}, P'_{ux} and P'_{uy} are pairwise vertex-disjoint except possibly at their endvertices.

If $v' \in K_1 - \{u\}$, then there exists a path P' in K_1 connecting u' and v' such that P' is vertex-disjoint from at least one of P_{ux} and P_{uy}, say P_{ux}. Then $uP_{ux}xQv$ and $u'P'v'$ are two vertex-disjoint paths, where Q is a path in K_2 connecting x and v, a contradiction to Claim 2. Thus, $v' \in (K_2 - K_1) \cup \{u\}$.

Claim 5 *No two independent edges in G connecting $K_1 - K_2$ and $K_2 - K_1$ exist. In particular, $v' = u$ or $v' = v$.*

Proof. Suppose that there exist two independent edges in G connecting $K_1 - K_2$ and $K_2 - K_1$. Then by renaming if necessary, we can find two independent edges uv and $u'v'$ such that $u \in K_1 - K_2 - \partial H(\mathbf{A})$, $u' \in \partial H(\mathbf{A}) \cap K_1 - K_2$, and $v, v' \in K_2 - K_1$.

Since $K_2 + xy$ is 2-connected, there are two vertex-disjoint paths Q_1, Q_2 in K_2 from $\{v, v'\}$ to $\{x, y\}$. By symmetry, we may assume that Q_1 connects v and x, and Q_2 connects v' and y. Then $uP_{ux}xQ_1v$ and $u'P'_{uy}yQ_2v'$ are two vertex-disjoint paths, a contradiction to Claim 2.

Thus, no two independent edges in G connecting $K_1 - K_2$ and $K_2 - K_1$ exist. Since $v' \in (K_2 - K_1) \cup \{u\}$, we have that $v' = u$ or $v' = v$. □

Since $u \in K_1 - K_2 - \partial H(\mathbf{A})$ and $u' \in \partial H(\mathbf{A}) \cap K_1 - K_2$, $|K_1| \geq 4$. If $|K_2| \geq 5$, then by Claim 5, $(\widetilde{K_1}, K_2)$ is a 3-separation in G with $|\widetilde{K_1}|, |K_2| \geq 5$, where $\widetilde{K_1} = K_1 \cup \{v\}$. This contradicts that G is internally 4-connected. Therefore, we obtain $|K_2| \leq 4$. By symmetry, we may assume that $xv \in E(G)$. By Proposition 5, $K_2 \cap A = \emptyset$ for any $A \in \mathbf{A}$.

Claim 6 *There is no edge in $E(G) - E(H)$ connecting two vertices of K_1 except for xy. In particular, $v' = v$.*

Proof. Suppose that there is an edge u_3v_3 in $E(G) - E(H)$ connecting two vertices of K_1 except for xy. If both u_3 and v_3 are in $\partial H(\mathbf{A})$, then the path R along $\partial H(\mathbf{A}) - \{x, y\}$ between u_3 and v_3, and the path P_{ux} together with v, are vertex-disjoint, a contradiction to Claim 2.

Thus we may assume that $u_3 \notin V(\partial H(\mathbf{A}))$. By the symmetry, we may also assume that $u \neq v_3$, and notice that $v \neq u_3, v_3$.

Suppose that $u \neq u_3$. By Claim 2, there are no two vertex-disjoint paths P_1, P_2 such that P_1 joins u and v, and P_2 joins u_3 and v_3. Thus by Proposition 6, $H(\mathbf{A})$ has a 2-separation (K_1', K_2') such that $K_1' \cap K_2' \subset V(\partial H(\mathbf{A}))$, $u_3 \in V(K_1' - K_2')$, and $u, v, v_3 \in K_2'$. By the existence of the paths P_{ux}, P_{uy}, $x, y \in K_2'$ and K_1' does not contain any vertex in $K_2 - K_1$. It follows that $|K_2'| \geq 5$. Let us observe that the 2-separation (K_1', K_2') can be extended to a 2-separation (K_1'', K_2'') in H such that $K_1'' \cap K_2'' \subset V(\partial H(\mathbf{A}))$, $u_3 \in V(K_1'' - K_2'')$, and $u, v, v_3 \in K_2''$. In addition $|K_2''| \geq 5$. But then this contradicts our choice (K_1, K_2) for the minimality of K_1, because $u_3 \notin V(\partial H(\mathbf{A}))$.

Thus, $u = u_3$. If $v_3 \neq u'$, then considering two pairs (u', v') and (u_3, v_3) in the above argument instead of (u, v) and (u_3, v_3), we obtain the same contradiction. So, we have $v_3 = u'$, in particular, there exists no edge in $E(G) - E(H)$ connecting two vertices of K_1 except for xy and uu'.

If there exists a path P in $K_1 - \{x, y\}$ connecting u and u', then $uPu'u$ and $u_1Qv_1u_1$ are two vertex-disjoint odd cycles, where Q is a path in K_2 connecting u_1 and v_1, a contradiction. Therefore, there exists a 2-separation (F_1, F_2) in K_1 such that $u \in F_1$, $u' \in F_2$, and $F_1 \cap F_2 = \{x, y\}$.

Recall that there exists no edge in $E(G) - E(H)$ connecting two vertices of K_1 except for xy and uu'. If $|F_1| \geq 5$, it follows from the above fact and Claim 5 that $(F_1, \widetilde{K_2})$ is a 3-separation in G with $|F_1|, |\widetilde{K_2}| \geq 5$, a contradiction, where $\widetilde{K_2} = F_2 \cup K_2$. Therefore, $|F_1| \leq 4$, and by symmetry, $|F_2| \leq 4$. By Proposition 5, $F_1 \cap A = F_2 \cap A = \emptyset$ for any $A \in \mathbf{A}$. These imply that $|G| \leq 8$, which contradicts the inequality (1). This completes the proof of Claim 6. □

When $|K_2| = 4$, let v'' be the unique vertex in $K_2 - \{v, x, y\}$. Using the symmetry of K_2 and the fact that H is 2-connected and bipartite, we apply Claim 5 to get the following three cases;

(I) $|K_2| = 3$, $x = u_1$, and $y = v_1$, or

(II) $|K_2| = 4$, $x = u_1$, $y = v_1$, and $E(K_2) = \{u_1v, u_1v'', vv_1, v''v_1\}$, or

(III) $|K_2| = 4$, $x = u_1$, $v'' = v_1$, and $E(K_2) = \{u_1v, vv_1, v_1y\}$.

By Claim 5, $(N_G(v'') \cap K_1) - K_2 = \emptyset$, and hence $N_G(v'') = \{v, x, y\}$ since G is 3-connected.

Let V_1, V_2 be the partite sets of the bipartite graph H with $u \in V_1$. Note that $v \in V_1$ and $u_1, v_1 \in V_2$ by the choice of H and by the above construction of K_2.

In the case (I) or (II), we obtain $xy = u_1v_1$, and in the case (III), $xy \notin E(G) - E(H)$ because $x \in V_2$, $y \in V_1$. Therefore, by Claims 5 and 6, $G - \{vu_1, vv_1, u_1v_1\}$ is a bipartite graph on V_1' and V_2', where $V_1' = V_1 - \{v\}$ and $V_2' = V_2 \cup \{v\}$. Hence (ii) holds; otherwise $S \subset \partial H(\mathbf{A})$, and this completes the proof. □

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