Forbidden induced subgraphs for star-free graphs

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Abstract

Let $\mathcal{H}$ be a family of connected graphs. A graph G is said to be $\mathcal{H}$-free if G is H-free for every graph H in $\mathcal{H}$. In [1] it was pointed that there is a family of connected graphs $\mathcal{H}$ not containing any induced subgraph of the claw having the property that the set of $\mathcal{H}$-free connected graphs containing a claw is finite, provided also that those graphs have minimum degree at least two and maximum degree at least three. In the same work, it was also asked whether there are other families with the same property. In this paper we answer this question by solving a wider problem. We consider not only claw-free graphs but the more general class of star-free graphs. Concretely, given $t \geq 3$, we characterize all the graph families $\mathcal{H}$ such that every large enough $\mathcal{H}$-free connected graph is $K_{1,t}$-free. Additionally, for the case $t = 3$ we show the families that one gets when adding the condition $|\mathcal{H}| \leq k$ for each positive integer k.

Keywords: star-free, claw-free, forbidden subgraph

1 Introduction

In this paper we only consider simple finite graphs. Let G be a connected graph. Given a connected graph H, G is said to be H-free if G does not contain H as an induced subgraph. Given a family of connected graphs $\mathcal{H}$, G is said to be $\mathcal{H}$-free if G is H-free for all $H \in \mathcal{H}$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of G, respectively.

If we have several families of forbidden subgraphs implying some given property, it is important to compare them to understand which families lead to more general results. Concretely, if we have two families of graphs $\mathcal{F}_1$ and $\mathcal{F}_2$, and all $\mathcal{F}_1$-free graphs are also $\mathcal{F}_2$-free graphs, then we can say that $\mathcal{F}_2$ is more general, in the sense that a result that states that all $\mathcal{F}_2$-free graphs satisfy some property is more general than one that says that all $\mathcal{F}_1$-free graphs satisfy the same property.

To do such comparisons, one can define some notion of order between forbidden families of graphs. The usual order used is to define that a family $\mathcal{F}_2$ is bigger than another family $\mathcal{F}_1$ if for each graph H in $\mathcal{F}_2$, there is a graph in $\mathcal{F}_1$ that is an induced subgraph of H. But the authors of [1] showed that sometimes a simple comparison by "inclusion of graphs" between

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families might not be enough. Consider the following theorem about graphs having a 2-factor (see Sections 2 and 4 for graph definitions). Remember that a 2-factor is a spanning subgraph with every vertex having degree two.

Theorem 1 ([1])**.** *Let G be a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$.*

(i) If G is $\{Z_{2,3}, K_{1,3}\}$-free then G has a 2-factor.

(ii) If G is $\{Z_{2,3}, Y_4, W_2^3, K_{2,3}\}$-free and $|V(G)| \geq 9$ then G has a 2-factor.

Because $Z_{2,3}$ is an induced subgraph of it self, and all three graphs Y_4, W_2^3 and $K_{2,3}$ contain a $K_{1,3}$ as an induced subgraph, we can say that (ii) is more general than (i). But on the other hand, we have the following result.

Theorem 2 ([1])**.** *Let G be a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$. If G is $\{Z_{2,3}, Y_4, W_2^3, K_{2,3}\}$-free and $|V(G)| \geq 9$, then G is $K_{1,3}$-free.*

Theorem 2 says that $\{Z_{2,3}, K_{1,3}\}$-free graphs and $\{Z_{2,3}, Y_4, W_2^3, K_{2,3}\}$-free are essentially (under some conditions) the same. This is not clear just by looking at the graphs in the families.

Another interesting point about Theorem 2 is that even though no graph of the family $\mathcal{H} = \{Z_{2,3}, Y_4, W_2^3, K_{2,3}\}$ is an induced subgraph of $K_{1,3}$, when considering the $\mathcal{H}$-free graphs under certain conditions, the graph $K_{1,3}$ is also forbidden. The authors of [1] were interested in finding a family of forbidden subgraphs implying a 2-factor that does not contain a star. But even though there is no star in $\{Y_4, Z_{2,3}, W_2^3, K_{2,3}\}$, by Theorem 2 it is somehow implicitly forbidden. That is why the authors of [1] called this phenomenon *implicit forbiddance*.

In the view of the previous results, in order to get more information of the implicit relation between families of forbidden subgraphs, it is important to research further this phenomenon. As a first step, we consider the case of $K_{1,3}$-free graphs, also in an effort to try to extend Theorem 2. We do so also because claw-free graphs have been widely studied in the literature, as they are closely related to line graphs, and on the other side, there are many interesting results in connection with matching theory and hamiltonian graph theory (see for example [6] for a survey on claw-free graphs). In this paper, we actually consider a more general class, star-free graphs. Concretely, we consider the following problem. Given $t \geq 3$, characterize all the families of connected graphs $\mathcal{H}$ such that every large enough $\mathcal{H}$-free connected graph is $K_{1,t}$-free. In this paper, we solve this problem for every $t \geq 3$.

The rest of the paper is organized as follows. In Section 2, we make all needed definitions and present our main results. In Section 3, we give the proofs for those results. In Section 4, we consider restricting the size of the family of forbidden subgraphs for the case $t = 3$. Concretely, we characterize the families of forbidden subgraphs $\mathcal{H}$ with $|\mathcal{H}| \leq k$ for each $k \geq 1$. See Section 4 for a formal statement of the problem. In Section 5, we give the proofs for the theorems presented in Section 4. In Section 6, we show an application of our results. Finally, in Section 7 we make some discussion, propose some open questions and comment on the cases $t = 1$ and $t = 2$.

2 Definitions and main results

If H_1 and H_2 are two connected graphs, we write $H_1 \preceq H_2$ to indicate that H_1 is an induced subgraph of H_2. We say that a family of connected graphs $\mathcal{H}$ is *redundant* if there are two different graphs $H_1, H_2 \in \mathcal{H}$ such that $H_1 \preceq H_2$. It is easy to see that we can restrict our problem to considering only non-redundant families.

Define $\mathbf{G}$ as the set of all non-redundant families of connected graphs. Let $t \geq 3$ and define $\mathbf{H}(t)$ as the set of families $\mathcal{H} \in \mathbf{G}$ such that there is a constant $n_0 = n_0(t, \mathcal{H})$ with the property that all $\mathcal{H}$-free connected graphs G with $|V(G)| \geq n_0$ are $K_{1,t}$-free. Then, our problem is reduced to finding all the elements in the set $\mathbf{H}(t)$.

We define a binary relation "$\leq$" in $\mathbf{G}$ as follows. For $\mathcal{H}_1$, $\mathcal{H}_2 \in \mathbf{G}$, we say that $\mathcal{H}_1 \leq \mathcal{H}_2$ if for each $H_2 \in \mathcal{H}_2$, there is an $H_1 \in \mathcal{H}_1$ such that $H_1 \preceq H_2$. It is easy to see that the relation "$\leq$" defines a partial order in $\mathbf{G}$. Furthermore, if $\mathcal{H}_1 \leq \mathcal{H}_2$ then any $\mathcal{H}_1$-free graph is also an $\mathcal{H}_2$-free graph (see for example Lemma 3 of [10]).

K_n is the complete graph on n vertices. P_n is the path on n vertices. $K_{n,m}$ is the complete bipartite graph with partite sets on n and m vertices.

Let $t \geq 2$. To state our results we define the following graphs (see Figure 1).

- Y^t_m is a path on m vertices with $t-1$ extra vertices attached to the first vertex of the path. The last vertex of the path is called the tail of Y^t_m. ($m \geq 1$)
- $Y^t_{s,m}$ is the graph obtained by joining s degree one vertices of a $K_{1,t}$ with the first vertex of the path on m vertices. The last vertex of the path is called the tail of $Y^t_{s,m}$. ($m \geq 1, 1 \leq s \leq t$)
- $\widehat{Y}^t_{s,m}$ is the graph obtained by joining s degree one vertices of a $K_{1,t}$ with the first vertex of the path on m vertices and adding the edge between the center of the $K_{1,t}$ and the first vertex of the path. The last vertex of the path is called the tail of $\widehat{Y}^t_{s,m}$. ($m \geq 1, 1 \leq s \leq t$)
- W^t_q is the graph obtained by completely joining a K_q with t independent vertices. ($q \geq 1$)
- $T^t_{s,q}$ is the graph obtained by joining s degree one vertices of a $K_{1,t}$ with all the vertices of a K_q. ($q \geq 1, 1 \leq s \leq t$)
- $D^t_{s,q}$ is the graph obtained by joining s degree one vertices and the center of a $K_{1,t}$ with all the vertices of a K_q. ($q \geq 1, 0 \leq s \leq t$)
- $Z^t_{m,r}, Z^t_{s,m,r}$ and $\widehat{Z}^t_{s,m,r}$ are the graphs obtained by identifying a vertex of a K_r with the tail of a $Y^t_m, Y^t_{s,m}$ and $\widehat{Y}^t_{s,m}$, respectively. ($m \geq 1, r \geq 1, 1 \leq s \leq t$)

For $t \geq 3$, define the following families of graphs.

- $\mathcal{T}^t(q) = \{\ T^t_{s,q}\colon\ 2 \leq s \leq t-1\ \}$.
- $\mathcal{D}^t(q) = \{\ D^t_{s,q}\colon\ 2 \leq s \leq t-1\ \}$.
- $\mathcal{Y}^t(m) = \{\ Y^t_{s,m}\colon\ 2 \leq s \leq t-2\ \}$.
- $\mathcal{Z}^t(m,r) = \{\ Z^t_{s,m,r}\colon\ 2 \leq s \leq t-2\ \}$.
- $\widehat{\mathcal{Y}}^t(m) = \{\ \widehat{Y}^t_{s,m}\colon\ 2 \leq s \leq t-2\ \}$.
- $\widehat{\mathcal{Z}}^t(m,r) = \{\ \widehat{Z}^t_{s,m,r}\colon\ 2 \leq s \leq t-2\ \}$.
- $\mathcal{YZ}^t(m,r) = \mathcal{Y}^t(m+2) \cup \mathcal{Z}^t(1,r) \cup \ldots \cup \mathcal{Z}^t(m,r) \cup \widehat{\mathcal{Y}}^t(m+2) \cup \widehat{\mathcal{Z}}^t(1,r) \cup \ldots \cup \widehat{\mathcal{Z}}^t(m,r)$.
- $\mathcal{H}^t(m,l,q,r) = \{K_{1,l}, W^t_q\} \cup \{Y^t_{m+2}, Z^t_{1,r}, \ldots, Z^t_{m,r}\} \cup \mathcal{T}^t(q) \cup \mathcal{D}^t(q) \cup \mathcal{YZ}^t(m,r)$.

Notice that for the case $t = 3$, $\mathcal{Y}^t(m)$, $\mathcal{Z}^t(m,r)$, $\widehat{\mathcal{Y}}^t(m)$ and $\widehat{\mathcal{Z}}^t(m,r)$ are empty and both $\mathcal{T}^t(q)$ and $\mathcal{D}^t(q)$ have only one element.

For $t \geq 3$, define the following subset of $\mathbf{G}$.

- $\mathbf{F}(t) = \{\ \mathcal{H} \in \mathbf{G}\colon\ \mathcal{H} \leq \mathcal{H}^t(m,l,q,r)$ for some $m \geq 1,\ l \geq t+1,\ q \geq 2,\ r \geq 3\}$.

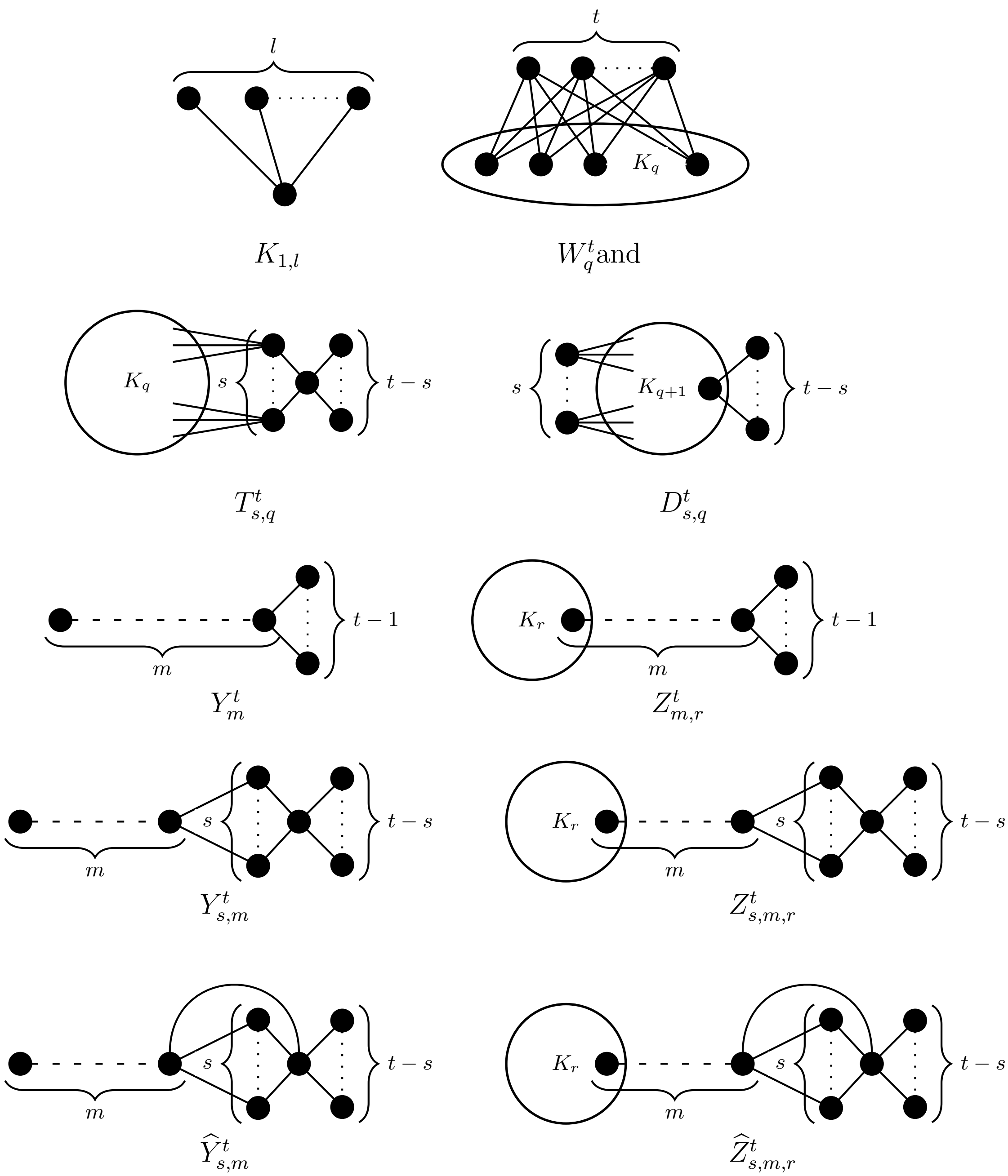

Figure 1: Some forbidden subgraphs

Our main result in this paper is the following theorem. It gives the characterization of families of forbidden subgraphs for star-free graphs we described in Section 1.

Theorem 3. *Let* $t \geq 3$*, then* $\mathbf{H}(t) = \mathbf{F}(t)$*.*

For our proofs we need the following definitions. For terminology and notation not defined in this paper, we refer the reader to [4].

Let G be a connected graph. For $v \in V(G)$, define $N_G^i(v) = \{w \in V(G)$: the distance from v to w is exactly $i\}$. Notice that $N_G^0(v) = \{v\}$ and $N_G^1(v) = N_G(v)$. If the graph G is obvious from the context, we sometimes write $N^i(v)$ for $N_G^i(v)$.

A *clique* of a graph is a set of pairwise adjacent vertices, and an *independent set* is a set of pairwise nonadjacent vertices. For two positive integers l and r, the *Ramsey number* $R(l, r)$ is the minimum positive integer R such that any graph of order at least R contains either an independent set of cardinality l or a clique of cardinality r. The Ramsey number $R(l, r)$ exists for every positive integers l and r (see for example [4]).

If G is a graph and $S \subseteq V(G)$, for $S' \subseteq S$, define $B_S(S') = \{v \in V(G) : N(v) \cap S = S'\}$.

Observation: if for some $N \subseteq V(G)$, there is a constant k such that for every $S' \subseteq S$, $|N \cap B_S(S')| \leq k$, then $|N| \leq 2^{|S|} \cdot k$ (remember that the number of subsets of a set S is $2^{|S|}$). We will implicitly use this fact in the proofs of several lemmas in section 3.

3 Proof of Theorem 3

First, we will prove the following theorem that shows that forbidding some family of $\mathbf{F}(t)$ is enough to imply that the graph is star-free provided it is large enough.

Theorem 4. *Let $t \geq 3$. Then $\mathbf{F}(t) \subseteq \mathbf{H}(t)$.*

Before giving the proof, we would like to comment on non-redundancy of the family $\mathcal{H}^t(m, l, q, r)$. It is not difficult to check that the family $\mathcal{H}^t(m, l, q, r)$ is non-redundant for the parameters used in the definition of $\mathbf{F}(t)$ ($m \geq 1$, $l \geq t + 1$, $q \geq 2$, $r \geq 3$). These conditions were chosen so that $\mathcal{H}^t(m, l, q, r)$ is not redundant nor it contains any induced subgraph of $K_{1,t}$. Moreover, reducing by 1 any of the constants in the condition would make $\mathcal{H}^t(m, l, q, r)$ either redundant or contain an induced subgraph of $K_{1,t}$. For example, if $q = 1$ then for all $m \geq 1$ and all $1 \leq s \leq t$ we have that $T^t_{s,q} \preceq Y^t_{s,m}$ and $T^t_{s,q} \preceq Z^t_{s,m}$; additionally W^t_q is a $K_{1,t}$.

We divide the proof of Theorem 4 in several lemmas that we state and prove bellow.

Lemma 5. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. If G is $(\{Y^t_m\} \cup \mathcal{Y}^t(m) \cup \widehat{\mathcal{Y}}^t(m))$-free for some $m \geq 3$, then $N^{m+1}(x_0) = \emptyset$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$, such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Suppose that $N^{m+1}(x_0) \neq \emptyset$. We will show that G contains a Y^t_m, some graph of $\mathcal{Y}^t(m)$ or some graph of $\widehat{\mathcal{Y}}^t(m)$, which is a contradiction.

Let $k = m + 1$ and let $P = x_0 x_1 \cdots x_k$ be an induced path of G with $x_i \in N^i(x_0)$ for all $0 \leq i \leq k$. Notice that $N^j(x_0) \cap N(Y) = \emptyset$ for all $3 \leq j \leq k$. Otherwise, an element $v \in N^j(x_0) \cap N(Y)$ would have a path of length 2 to x_0, passing through some element of Y, contradicting that $v \in N^j(x_0)$. Then $N(Y) \cap P \subseteq \{x_0, x_1, x_2\}$.

Let $Y_1 = N(x_1) \cap Y$ and $Y_2 = N(x_2) \cap Y$. If $|Y_2| \geq t - 1$, then $Y_2 \cup \{x_2, \ldots, x_{m+1}\}$ contains a Y^t_m. If $2 \leq |Y_2| \leq t - 2$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m+1}\}$ is a $Y^t_{s,m}$, where $s = |Y_2|$. If $|Y_2| = 1$, then $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_{m-1}\}$ is a Y^t_m.

Suppose now that $|Y_2| = 0$, that is $N(x_2) \cap Y = \emptyset$. If $|Y_1| \geq t - 1$, then $Y_1 \cup \{x_1, \ldots, x_m\}$ contains a Y^t_m. If $2 \leq |Y_1| \leq t - 2$, then $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1, \ldots, x_m\}$ is a $\widehat{Y}^t_{s,m}$, where $s = |Y_1|$. If $|Y_1| \leq 1$, then $(Y - Y_1) \cup \{x_0, \ldots, x_{m-1}\}$ contains a Y^t_m. □

Lemma 6. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. Suppose that G is $(\{K_{1,l}, Z^t_{1,r}, W^t_q\} \cup \mathcal{D}^t(q))$-free for some $l \geq t + 1$, $r \geq 3$, $q \geq 2$. Then $|N(x_0)| < 2^t \cdot R(l, \max(r, q))$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$, such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Let $Y' \subseteq Y$. We will show that $|N(x_0) \cap B_Y(Y')| < R(l, \max(r, q))$, and since $|Y| = t$ we get that $|N(x_0)| < 2^t \cdot R(l, \max(r, q))$.

If $|Y'| \leq 1$, then $|Y - Y'| \geq t - 1$ and so $|N(x_0) \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y') \cup \{x_0\} \cup (N(x_0) \cap B_Y(Y'))$ contains a $Z^t_{1,r}$ or a $K_{1,l}$.

If $2 \leq |Y'| \leq t - 1$, then $|N(x_0) \cap B_Y(Y')| < R(l, q)$, since otherwise $Y' \cup (Y - Y') \cup \{x_0\} \cup (N(x_0) \cap B_Y(Y'))$ contains a $D^t_{s,q}$ or a $K_{1,l}$, where $s = |Y'|$.

If $|Y'| = t$, then $|N(x_0) \cap B_Y(Y')| < R(l, q)$, since otherwise $Y' \cup (N(x_0) \cap B_Y(Y'))$ contains a W^t_q or a $K_{1,l}$. $\square$

Lemma 7. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. Suppose that G is $(\{K_{1,l}, Z^t_{1,r}, Z^t_{2,r}, W^t_q\} \cup \widehat{\mathcal{Z}}^t(1, r) \cup \mathcal{T}^t(q))$-free for some $l \geq t + 1$, $r \geq 3$, $q \geq 2$. Then $|N^2(x_0)| < 2^t \cdot R(l, \max(r, q)) \cdot |N(x_0)|$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$, such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Let $x_1 \in N(x_0)$. Let $Y' \subseteq Y$. Let $N = N^2(x_0) \cap N(x_1)$. It suffices to show that $|N \cap B_Y(Y')| < R(l, \max(r, q))$.

If $|Y'| = 1$, then $|Y - Y'| = t - 1$ and so $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y') \cup \{x_0\} \cup Y' \cup (N \cap B_Y(Y'))$ contains a $Z^t_{2,r}$ or a $K_{1,l}$.

If $2 \leq |Y'| \leq t - 1$, then $|N \cap B_Y(Y')| < R(l, q)$, since otherwise $(Y - Y') \cup \{x_0\} \cup Y' \cup (N \cap B_Y(Y'))$ contains a $T^t_{s,q}$ or a $K_{1,l}$, where $s = |Y'|$.

If $|Y'| = t$, then $|N \cap B_Y(Y')| < R(l, q)$, since otherwise $Y' \cup (N \cap B_Y(Y'))$ contains a W^t_q or a $K_{1,l}$.

Suppose now that $|Y'| = 0$, that is $N \cap B_Y(Y') \cap N(Y) = \emptyset$. Notice that if $x_1 \in Y$, then $N \cap B_Y(Y') = \emptyset$. Then we may suppose that $x_1 \notin Y$. Let $Y_1 = Y \cap N(x_1)$.

If $|Y_1| \geq t - 1$, then $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $Y_1 \cup \{x_1\} \cup (N \cap B_Y(Y'))$ contains a $Z^t_{1,r}$ or a $K_{1,l}$.

If $2 \leq |Y_1| \leq t - 2$, then $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1\} \cup (N \cap B_Y(Y'))$ contains a $\widehat{Z}^t_{s,1,r}$ or a $K_{1,l}$, where $s = |Y_1|$.

If $|Y_1| \leq 1$, then $|Y - Y_1| \geq t - 1$ and so $|N \cap B_Y(Y')| < R(l, r)$, since otherwise $(Y - Y_1) \cup \{x_0, x_1\} \cup (N \cap B_Y(Y'))$ contains a $Z^t_{2,r}$ or a $K_{1,l}$. $\square$

Lemma 8. *Let $t \geq 3$ and let G be a graph with an induced $K_{1,t}$ of center x_0. Let $i \geq 2$ and suppose that G is $(\{K_{1,l}, Z^t_{i-1,r}, Z^t_{i,r}, Z^t_{i+1,r}\} \cup \mathcal{Z}^t(i - 1, r) \cup \widehat{\mathcal{Z}}^t(i, r))$-free for some $l \geq t + 1$ and $r \geq 3$. Then $|N^{i+1}(x_0)| < R(l, r) \cdot |N^i(x_0)|$.*

Proof. Let $Y \subseteq V(G)$ with $|Y| = t$, such that $\{x_0\} \cup Y$ is an induced $K_{1,t}$ in G. Let $x_i \in N^i(x)$ and let $x_0 x_1 \cdots x_i$ be an induced path with $x_j \in N^j(x)$ for all $0 \leq j \leq i$. Let $N = N^{i+1}(x_0) \cap N(x_i)$. It suffices to show that $|N| < R(l, r)$.

Let $Y_1 = Y \cap N(x_1)$ and $Y_2 = Y \cap N(x_2)$. As in the proof of Lemma 5, for all $3 \leq j \leq i+1$, $N^j(x) \cap N(Y) = \emptyset$.

If $|Y_2| \geq t - 1$, then $|N| < R(l, r)$, since otherwise $Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z^t_{i-1,r}$ or a $K_{1,l}$.

If $2 \leq |Y_2| \leq t-2$, then $|N| < R(l, r)$, since otherwise $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z^t_{s,i-1,r}$ or a $K_{1,l}$, where $s = |Y_2|$.

If $|Y_2| = 1$, then $|N| < R(l, r)$, since otherwise $(Y - Y_2) \cup \{x_0\} \cup Y_2 \cup \{x_2, \ldots, x_i\} \cup N$ contains a $Z^t_{i+1,r}$ or a $K_{1,l}$.

Suppose now that $|Y_2| = 0$, that is $N(x_2) \cap Y = \emptyset$.

If $|Y_1| \geq t - 1$, then $|N| < R(l, r)$, since otherwise $Y_1 \cup \{x_1, \ldots, x_i\} \cup N$ contains a $Z^t_{i,r}$ or a $K_{1,l}$.

If $2 \leq |Y_1| \leq t-2$, then $|N| < R(l, r)$, since otherwise $(Y - Y_1) \cup \{x_0\} \cup Y_1 \cup \{x_1, \ldots, x_i\} \cup N$ contains a $\widehat{Z}^t_{s,i,r}$ or a $K_{1,l}$, where $s = |Y_1|$.

If $|Y_1| \leq 1$, then $|Y - Y_1| \geq t - 1$ and so $|N| < R(l, r)$, since otherwise $(Y - Y_1) \cup \{x_0, \ldots, x_i\} \cup N$ contains a $Z^t_{i+1,r}$ or a $K_{1,l}$. $\square$

We use the above lemmas to prove Theorem 4.

Proof of Theorem 4. Let $\mathcal{H} \in \mathbf{F}(t)$. Let $m \geq 1$, $l \geq t + 1$, $q \geq 2$ and $r \geq 3$ such that $\mathcal{H} \leq \mathcal{H}^t(m, l, q, r)$.

Let G be an $\mathcal{H}$-free connected graph. Suppose that there is an induced $K_{1,t}$ of center x_0. We will show that $|V(G)|$ is bounded by a function depending only on t, l, m, q and r.

Notice that since G is Y^t_{m+2}-free, then G is $Z^t_{i,r}$-free for all $i \geq m+1$. Since we also know that G is $Z^t_{i,r}$-free for all $1 \leq i \leq m$, we conclude that G is $Z^t_{i,r}$-free for all $i \geq 1$. Using a similar argument, we have that G is $\mathcal{Z}^t(i,r)$-free and $\widehat{\mathcal{Z}}^t(i,r)$-free for all $i \geq 1$. Thus, G satisfies all the conditions of Lemmas 5, 6, 7 and 8.

By Lemma 5, $N^{m+1}(x) = \emptyset$. Then we only need to show that $N^i(x)$ is bounded for all $1 \leq i \leq m$. By Lemmas 6 and 7, $N(x)$ and $N^2(x)$ are bounded. By Lemma 8, $|N^{i+1}(x)| < R(l,r) \cdot |N^i(x)|$ for all $2 \leq i \leq m-1$. Using an inductive argument we get that $|N^i(x)| < R(l,r)^{i-2} \cdot |N^2(x)|$ for all $3 \leq i \leq m$. We conclude that $|N^i(x)| < R(l,r)^{m-2} \cdot |N^2(x)|$ for all $3 \leq i \leq m$. □

Finally, we prove our main theorem.

Proof of Theorem 3. By Theorem 4, we already know that every family of graphs in $\mathbf{F}(t)$ is also in $\mathbf{H}(t)$. It remains to show that every family of graphs in $\mathbf{H}(t)$ is also in $\mathbf{F}(t)$.

Let $\mathcal{H} \in \mathbf{H}(t)$. Then there is a positive integer n_0 such that every $\mathcal{H}$-free connected graph of order at least n_0 is $K_{1,t}$-free. Let n be an integer such that $n \geq \max(n_0, t+1)$.

Consider the family $\mathcal{H}' = \mathcal{H}^t(n,n,n,n)$. All the graphs in $\mathcal{H}'$ are connected graphs of order at least n_0 containing an induced $K_{1,t}$. Then it must be that no graph of $\mathcal{H}'$ is $\mathcal{H}$-free. In other words, for each $H' \in \mathcal{H}'$, there is an $H \in \mathcal{H}$ such that $H \preceq H'$. But this is exactly the definition of $\mathcal{H} \leq \mathcal{H}'$. Then since $\mathcal{H}'$ is in $\mathbf{F}(t)$, we conclude that $\mathcal{H}$ is also in $\mathbf{F}(t)$. □

4 Restricting the size of the family of forbidden subgraphs

When searching for families of forbidden subgraphs implying some property on graphs, it is usual to start by restricting the size of the family of forbidden subgraphs. The reason for that is that it makes easier to deal with the problem and it also provides partial but self-contained results. See [3, 7, 8, 9, 12, 13] for examples of papers using this technique.

In this section, we show the families that we obtained for each possible size of the family.

Concretely, we add the condition $|\mathcal{H}| \leq k$ to some family $\mathcal{H} \in \mathbf{H}(t)$ for some positive integer k. We restrict ourselves to the case $t = 3$ (claw-free graphs) which has been widely studied in the literature. We characterize such families for each $k \geq 1$. In other words, for each $k \geq 1$, we characterize the families $\mathcal{H} \in \mathbf{H}(3)$ such that $|\mathcal{H}| \leq k$. The result of the characterization is expressed in Theorem 9.

To state and prove the result, we do some notation changes in graph names to reduce the number of subindices and superindices.

- Y_m is Y^3_m.
- $Z_{m,r}$ is $Z^3_{m,r}$.
- D_q is $D^3_{2,q}$.
- T_q is $T^3_{2,q}$.

We also define some additional graphs.

- $Z^-_{m,r}$ is the graph obtained by identifying a vertex of a K_r with the end vertex of a path on $m+1$ vertices.
- T^-_q is T_q with the only vertex of degree one of T_q removed.

Define the following families of graphs.

- $\mathcal{H}^A_i(l,q,r) = \{K_{1,l}, Y_{i+2}, W^2_q, Z_{1,r}, \ldots, Z_{i,r}\}$ (for $i \geq 1$).

- $\mathcal{H}_i^B(l,m,q,r) = \{K_{1,l}, Y_m, W_q^2, Z_{1,r}, \ldots, Z_{i-1,r}, Z_{i,r}^-\}$ (for $i \geq 2$).
- $\mathcal{H}_i^C(l,q,r) = \{K_{1,l}, Y_{i+2}, W_q^3, D_q, T_q, Z_{1,r}, \ldots, Z_{i,r}\}$ (for $i \geq 1$).
- $\mathcal{H}_i^D(l,m,q,r) = \{K_{1,l}, Y_m, W_q^3, D_q, T_q, Z_{1,r}, \ldots, Z_{i-1,r}, Z_{i,r}^-\}$ (for $i \geq 3$).

Define the following subsets of $\mathbf{G}$.

- $\mathbf{F}_1 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,3}\}\ \}$.
- $\mathbf{F}_3 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, K_r\}$ for some $l \geq 4$, $m \geq 3$ and $r \geq 3\}$.
- $\mathbf{F}_4 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, W_q^3, Z_{1,r}^-\}$ for some $l \geq 4$, $m \geq 3$, $q \geq 2$ and $r \geq 3\}$.
- $\mathbf{F}_5 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, P_4, W_q^3, D_q, Z_{1,r}\}$ for some $l \geq 4$, $q \geq 2$ and $r \geq 3\}$.
- $\mathbf{F}_6 = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \{K_{1,l}, Y_m, W_q^3, D_q, Z_{1,r}, Z_{2,r}^-\}$ for some $l \geq 4$, $m \geq 4$, $q \geq 2$ and $r \geq 3\}$.
- $\mathbf{F}_i^A = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}_i^A(l,q,r)$ for some $l \geq 4$, $q \geq 2$, $r \geq 3\}$ $(i \geq 1)$.
- $\mathbf{F}_i^B = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}_i^B(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$, $r \geq 3\}$ $(i \geq 2)$.
- $\mathbf{F}_i^C = \{\ \mathcal{H} \in \mathbf{G}$: $\mathcal{H} \leq \mathcal{H}_i^C(l,q,r)$ for some $l \geq 4$, $q \geq 2$, $r \geq 3\}$ $(i \geq 1)$.
- $\mathbf{F}_i^D = \{\ \mathcal{H} \in \mathbf{G} : \mathcal{H} \leq \mathcal{H}_i^D(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$, $r \geq 3\}$ $(i \geq 3)$.

The following is the main theorem of this section.

Theorem 9. *Let $k \geq 1$ be an integer and let $\mathcal{H} \in \mathbf{H}(3)$ with $|\mathcal{H}| \leq k$. Then*

- *$\mathcal{H} \in \mathbf{F}_i$ for some $i \in \{1,3,4,5,6\}$ with $i \leq k$ or*
- *$\mathcal{H} \in \mathbf{F}_i^A$ for some $1 \leq i \leq k-3$ or*
- *$\mathcal{H} \in \mathbf{F}_i^B$ for some $2 \leq i \leq k-3$ or*
- *$\mathcal{H} \in \mathbf{F}_i^C$ for some $1 \leq i \leq k-5$ or*
- *$\mathcal{H} \in \mathbf{F}_i^D$ for some $3 \leq i \leq k-5$.*

Notice that $\mathbf{F}(3) = \bigcup_{i \geq 1} \mathbf{F}_i^C$. Moreover, all the families in the other sets mentioned in Theorem 9 are also in $\mathbf{F}(3)$, and so they are all in $\mathbf{H}(3)$. This fact can by derived from the following lemma.

Lemma 10. *The following statements hold:*

(1) $\mathbf{F}_1 \subseteq \mathbf{F}_1^C$, $\mathbf{F}_3 \subseteq \mathbf{F}_4$, $\mathbf{F}_4 \subseteq \mathbf{F}_6$, $\mathbf{F}_5 \subseteq \mathbf{F}_1^C$ *and* $\mathbf{F}_6 \subseteq \mathbf{F}_3^D$.

(2) *Let $i \geq 1$, then $\mathbf{F}_i^A \subseteq \mathbf{F}_i^C$.*

(3) *Let $i \geq 2$, then $\mathbf{F}_i^B \subseteq \mathbf{F}_j^A$ for some $j \geq 1$.*

(4) *Let $i \geq 3$, then $\mathbf{F}_i^D \subseteq \mathbf{F}_j^C$ for some $j \geq 1$.*

Proof. Statements (1) and (2) are easy to verify.

Proof of (3): Let $i \geq 2$ and $\mathcal{H} \in \mathbf{F}_i^B$. Since $\mathcal{H} \leq \mathcal{H}_i^B(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$ and $r \geq 3$, we have that $\mathcal{H} \leq \{Y_m\}$ for some $m \geq i+3$. Since $Z_{i,r}^- \preceq Z_{h,r}$ for all $h \geq i$ and all $r \geq 3$, then $\mathcal{H} \in \mathbf{F}_{m-2}^A$.

Proof of (4): Let $i \geq 3$ and $\mathcal{H} \in \mathbf{F}_i^D$. Since $\mathcal{H} \leq \mathcal{H}_i^D(l,m,q,r)$ for some $l \geq 4$, $m \geq i+3$, $q \geq 2$ and $r \geq 3$, we have that $\mathcal{H} \leq \{Y_m\}$ for some $m \geq i+3$. Since $Z_{i,r}^- \preceq Z_{h,r}$ for all $h \geq i$ and all $r \geq 3$, then $\mathcal{H} \in \mathbf{F}_{m-2}^C$. □

5 Proof of Theorem 9

First, we prove two lemmas that deal with the inductive part of the proof of Theorem 9.

Lemma 11. *Let $k \geq 4$ be an integer and let $\mathcal{H} \in \mathbf{H}(3)$ with $|\mathcal{H}| \leq k$. Suppose that $\mathcal{H} \not\leq \{K_{1,3}\}$, $\mathcal{H} \notin \mathbf{F}^A_j$ for all $1 \leq j \leq k-3$ and $\mathcal{H} \notin \mathbf{F}^B_j$ for all $2 \leq j \leq k-3$. Suppose also that there are graphs $B_1, B_2, B_3, H_1 \in \mathcal{H}$ such that*

- *$B_1 = K_{1,l}$ for some $l \geq 4$.*
- *$B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 3$.*
- *$B_3 = W^2_q$ for some $q \geq 2$.*
- *$H_1 = Z_{1,r_1}$ for some $r_1 \geq 3$.*

Then there are graphs $H_2, \dots, H_{k-3}$ in $\mathcal{H}$ and integers $r_2, \dots, r_{k-3}$ such that for all $2 \leq i \leq k-3$, $H_i = Z_{i,r_i}$ and $r_i \geq 3$. Additionally, $m \geq k$.

Proof. The proof is by induction on i.

Let $2 \leq i \leq k-3$ and suppose that there are graphs $H_1, \dots, H_{i-1}$ in $\mathcal{H}$ such that $H_j = Z_{j,r_j}$ for some $r_j \geq 3$ and all $1 \leq j \leq i-1$. We will prove that there is a graph $H_i \in \mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$.

Let $r' = \max(r_1, \dots, r_{i-1})$. Since $\mathcal{H} \leq \{K_{1,l}, W^2_q, Z_{1,r'}, \dots, Z_{i-1,r'}\}$ and $\mathcal{H} \notin \mathbf{F}^A_{i-1}$, then $\mathcal{H} \not\leq \{Y_{i+1}\}$. In particular, $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq i+2$.

Since $\mathcal{H} \leq \{K_{1,l}, Y_m, W^2_q, Z_{1,r'}, \dots, Z_{i-1,r'}\}$ and $\mathcal{H} \notin \mathbf{F}^B_i$, then $\mathcal{H} \not\leq \{Z^-_{i,r}\}$ for all $r \geq 3$.

Since $\mathcal{H} \in \mathbf{H}(3)$, there is a positive integer $n_0 = n_0(\mathcal{H})$ such that every $\mathcal{H}$-free connected graph of order at least n_0 is claw-free. Let $n = \max(n_0, 3)$.

Consider $G = Z_{i,n}$. Since G contains an induced claw, G must contain some graph in $\mathcal{H}$ as an induced subgraph. Since G contains neither $K_{1,4}$, P_{i+3} nor W^2_2 then $B_j \not\preceq G$ for all $j \in \{1,2,3\}$. Furthermore, since $Z_{j,3} \not\preceq G$ for all $1 \leq j \leq i-1$, then $H_j \not\preceq G$ for all $1 \leq j \leq i-1$. Then there must be some other graph $H_i \in \mathcal{H}$ such that $H_i \preceq G$.

Since $H_i \not\preceq K_{1,3}$, $H_i \not\preceq Y_{i+1}$ and that $H_i \not\preceq Z^-_{i,r}$ for all $r \geq 3$, then $H_i = Z_{i,r_i}$ for some $r_i \geq 3$. Notice that if $r_i = 2$, then it would contradict that $H_i \not\preceq Y_{i+1}$.

This concludes the inductive proof. We now prove that $m \geq k$. Let $i = k-3$. Let $r = \max(r_1, \dots, r_i)$. Suppose that $\mathcal{H} \leq \{Y_{i+2}\}$. Then $\mathcal{H} \leq \{K_{1,l}, Y_{i+2}, W^2_q, Z_{1,r}, \dots, Z_{i,r}\}$, and hence $\mathcal{H} \leq \mathcal{H}^A_i(l,q,r)$ (with $i = k-3$), a contradiction. We conclude that $\mathcal{H} \not\leq \{Y_{i+2}\} = \{Y_{k-1}\}$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k$. □

Lemma 12. *Let $k \geq 7$ be an integer and let $\mathcal{H} \in \mathbf{H}(3)$ with $|\mathcal{H}| \leq k$. Suppose that $\mathcal{H} \not\leq \{K_{1,3}\}$, $\mathcal{H} \notin \mathbf{F}^C_j$ for all $1 \leq j \leq k-5$ and $\mathcal{H} \notin \mathbf{F}^D_j$ for all $3 \leq j \leq k-5$. Suppose also that there are graphs $B_1, \dots, B_5, H_1, H_2 \in \mathcal{H}$ such that*

- *$B_1 = K_{1,l}$ for some $l \geq 4$.*
- *$B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 3$.*
- *$B_3 = W^3_{q_1}$ for some $q_1 \geq 2$.*
- *$B_4 = D_{q_2}$ for some $q_2 \geq 2$.*
- *$B_5 = T^-_{q_3}$ or $H_5 = T_{q_3}$ for some $q_3 \geq 1$ and*
- *$H_1 = Z_{1,r_1}$ for some $r_1 \geq 3$.*
- *$H_2 = Z_{2,r_1}$ for some $r_2 \geq 3$.*

Then there are graphs $H_3, \dots, H_{k-5}$ in $\mathcal{H}$ and integers $r_3, \dots, r_{k-5}$ such that for all $3 \leq i \leq k-5$, $H_i = Z_{i,r_i}$ and $r_i \geq 3$. Additionally, $m \geq k-2$.

Proof. The proof of this lemma is essentially the same as Lemma 11. The proof is by induction on i.

Let $3 \leq i \leq k-5$ and suppose that there are graphs $H_1, \ldots, H_{i-1}$ in $\mathcal{H}$ such that $H_j = Z_{j,r_j}$ for some $r_j \geq 3$ and all $1 \leq j \leq i-1$. We will prove that there is a graph $H_i \in \mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$.

Let $r' = \max(r_1, \ldots, r_{i-1})$. Since $\mathcal{H} \leq \{K_{1,l}, W^3_{q_1}, D_{q_2}, T_{q_3}, Z_{1,r'}, \ldots, Z_{i-1,r'}\}$ and $\mathcal{H} \notin \mathbf{F}^C_{i-1}$, then $\mathcal{H} \not\leq \{Y_{i+1}\}$. In particular, $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq i+2$.

Since $\mathcal{H} \leq \{K_{1,l}, Y_m, W^3_{q_1}, D_{q_2}, T_{q_3}, Z_{1,r'}, \ldots, Z_{i-1,r'}\}$ and $\mathcal{H} \notin \mathbf{F}^D_i$, then $\mathcal{H} \not\leq \{Z^-_{i,r}\}$ for all $r \geq 3$.

Let n_0 be as in Lemma 11. Let $n = \max(n_0, 3)$. Consider $G = Z_{i,n}$. Since G contains neither $K_{1,4}$, P_{i+3}, W^3_2, D_2, T^-_1 then $B_j \not\preceq G$ for all $j \in \{1,2,3,4,5\}$. Furthermore, since $Z_{j,3} \not\preceq G$ for all $1 \leq j \leq i-1$, then $H_j \not\preceq G$ for all $1 \leq j \leq i-1$. Then there must be some other graph $H_i \in \mathcal{H}$ such that $H_i \preceq G$.

Since $H_i \not\preceq K_{1,3}$, $H_i \not\preceq Y_{i+1}$ and that $H_i \not\preceq Z^-_{i,r}$ for all $r \geq 3$, then $H_i = Z_{i,r_i}$ for some $r_i \geq 3$. Notice that if $r_i = 2$, then it would contradict that $H_i \not\preceq Y_{i+1}$.

This concludes the inductive proof.

We now prove that $m \geq k-2$. Let $i = k-5$. Let $r = \max(r_1, \ldots, r_i)$. Suppose that $\mathcal{H} \leq \{Y_{i+2}\}$. Then $\mathcal{H} \leq \{K_{1,l}, Y_{i+2}, W^3_{q_1}, D_{q_2}, T_{q_3}, Z_{1,r}, \ldots, Z_{i,r}\}$, and hence $\mathcal{H} \leq \mathcal{H}^C_i(l,q,r)$ (with $i = k-5$), a contradiction. We conclude that $\mathcal{H} \not\leq \{Y_{i+2}\} = \{Y_{k-3}\}$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k-2$. □

Proof of Theorem 9.

Suppose that $\mathcal{H} \in \mathbf{H}(3)$ and $|\mathcal{H}| \leq k$. Contrary to the theorem, suppose that

- $\mathcal{H} \notin \mathbf{F}_i$ for all $i \in \{1,3,4,5,6\}$ with $i \leq k$,
- $\mathcal{H} \notin \mathbf{F}^A_i$ for all $1 \leq i \leq k-3$,
- $\mathcal{H} \notin \mathbf{F}^B_i$ for all $2 \leq i \leq k-3$,
- $\mathcal{H} \notin \mathbf{F}^C_i$ for all $1 \leq i \leq k-5$ and
- $\mathcal{H} \notin \mathbf{F}^D_i$ for all $3 \leq i \leq k-5$.

Since $\mathcal{H} \in \mathbf{H}(3)$, there is a positive integer $n_0 = n_0(\mathcal{H})$ such that every $\mathcal{H}$-free connected graph of order at least n_0 is claw-free. Let $n = \max(n_0, 3)$. We will consider several connected graphs G of order at least n containing an induced claw. By the definition of $\mathbf{H}(3)$, there will be some $H \in \mathcal{H}$ such that $H \preceq G$.

Consider $G = K_{1,n}$. Then there is a graph $B_1 \in \mathcal{H}$ such that $B_1 \preceq G$. Since $\mathcal{H} \notin \mathbf{F}_1$, then $\mathcal{H} \not\leq \{K_{1,3}\}$, and so $B_1 \not\preceq K_{1,3}$. We conclude that

- $B_1 = K_{1,l}$ for some $l \geq 4$.

Consider $G = Y_n$. Since G contains no $K_{1,4}$, then $B_1 \not\preceq G$. Then $k \geq 2$ and there is a graph $B_2 \in \mathcal{H}$ such that $B_2 \preceq G$. Since $B_2 \not\preceq K_{1,3}$ then

- $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq 3$.

Consider $G = W^3_n$. Since G contains neither $K_{1,4}$ nor P_4, then $B_1 \not\preceq G$ and $B_2 \not\preceq G$. Then $k \geq 3$ and there is a graph $B_3 \in \mathcal{H}$ such that $B_3 \preceq G$. Since $\mathcal{H} \notin \mathbf{F}_3$, then $\mathcal{H} \not\leq \{K_r\}$ for all $r \geq 3$. Since $B_3 \not\preceq K_{1,3}$ and $B_3 \not\preceq K_r$ for all $r \geq 3$, then

- $B_3 = W^2_{q_1}$ or $B_3 = W^3_{q_1}$ for some $q_1 \geq 2$.

Consider $G = Z_{1,n}$. Since G contains neither $K_{1,4}$, P_4 nor W^2_2, then $B_i \not\preceq G$ for all $i \in \{1,2,3\}$. Then $k \geq 4$ and there is a graph $H_1 \in \mathcal{H}$ such that $H_1 \preceq G$ (the name H_1 will be better understand later in the proof). Since $\mathcal{H} \notin \mathbf{F}_4$, then $\mathcal{H} \not\leq \{Z^-_{1,r}\}$ for all $r \geq 3$. Since $H_1 \not\preceq K_{1,3}$ and $H_1 \not\preceq Z^-_{1,r}$ for all $r \geq 3$, then

- $H_1 = Z_{1,r_1}$ for some $r_1 \geq 3$.

Case 1: $\mathcal{H} \leq \{W_q^2\}$ for some $q \geq 2$.

Since $\mathcal{H} \leq \{W_q^2\}$ for some $q \geq 2$ then there is a graph B' in $\mathcal{H}$ such that $B' \preceq W_q^2$ for some $q \geq 2$. Notice it may be that $B' = B_3$ or not. Since $B' \npreceq K_{1,3}$ and $B' \npreceq K_r$ for all $r \geq 3$, then $B' = W_q^2$ for some $q \geq 2$.

By Lemma 11, there are graphs $H_2, \ldots, H_{k-3}$ in $\mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$ and all $2 \leq i \leq k-3$. From the same lemma, we have that $m \geq k$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k$. Notice that $\{B_1, B_2, B', H_1, \ldots, H_{k-3}\} \subseteq \mathcal{H}$. Since $|H| \leq k$, then $B' = B_3$ and $\mathcal{H}$ has no other graphs, namely, $\mathcal{H} = \{B_1, B_2, B_3, H_1, \ldots, H_{k-3}\}$.

Consider $G = Z_{k-2,n}$. Since G contains neither $K_{1,4}$, P_{k+1} nor W_2^2 then $B_i \npreceq G$ for all $i \in \{1,2,3\}$. Furthermore, since $Z_{i,3} \npreceq G$ for all $1 \leq i \leq k-3$, then $H_i \npreceq G$ for all $1 \leq i \leq k-3$. Then G contains no graph of $\mathcal{H}$, which is a contradiction.

Case 2: $\mathcal{H} \nleq \{W_q^2\}$ for all $q \geq 2$.

Since $\mathcal{H} \nleq \{W_q^2\}$ for all $q \geq 2$, then

- $B_3 = W_{q_1}^3$ for some $q_1 \geq 2$.

Consider $G = D_n$. Since G contains neither $K_{1,4}$, P_4, W_2^3 nor $Z_{1,3}$, then $B_i \npreceq G$ for all $i \in \{1,2,3\}$ and $H_1 \npreceq G$. Then $k \geq 5$ and there is a graph $B_4 \in \mathcal{H}$ such that $B_4 \preceq G$. Since $B_4 \npreceq K_{1,3}$, $B_4 \npreceq K_r$ for all $r \geq 3$, $B_4 \npreceq W_q^2$ for all $q \geq 2$ and $B_4 \npreceq Z_{1,r}^-$ for all $r \geq 3$, then

- $B_4 = D_{q_2}$ for some $q_2 \geq 2$.

Since $\mathcal{H} \notin \mathbf{F}_5$, then $\mathcal{H} \nleq \{P_4\}$. Then $B_2 = P_m$ for some $m \geq 5$, or $B_2 = Y_m$ for some $m \geq 3$.

Consider $G = T_n$. Since G contains neither $K_{1,4}$, P_5, Y_3, W_2^3, D_2 nor $Z_{1,3}$, then $B_i \npreceq G$ for all $1 \leq i \leq 4$ and $H_1 \npreceq G$. Then $k \geq 6$ and there is a graph $B_5 \in \mathcal{H}$ such that $B_5 \preceq G$. Since $\mathcal{H} \notin \mathbf{F}_6$, then $\mathcal{H} \nleq \{Z_{2,r}^-\}$ for all $r \geq 3$. Since $B_5 \npreceq K_{1,3}$, $B_5 \npreceq W_q^2$ for all $q \geq 2$, and that $B_5 \npreceq Z_{j,r}^-$ for $j \in \{1,2\}$ and all $r \geq 3$, then

- $B_5 = T_{q_3}^-$ or $B_5 = T_{q_3}$ for some $q_3 \geq 1$.

Suppose that $\mathcal{H} \leq \{Y_3\}$.
Since $\mathcal{H} \leq \{K_{1,l}, Y_3, W_{q_1}^3, D_{q_2}, T_{q_3}, Z_{1,r_1}\}$, then $\mathcal{H} \leq \mathcal{H}_1^C(l, \max(q_1,q_2,q_3), r_1)$, a contradiction (since $1 \leq k-5$). Then we may suppose that $\mathcal{H} \nleq \{Y_3\}$ and so $B_2 = P_{m+1}$, or $B_2 = Y_m$ for some $m \geq 4$.

Consider $G = Z_{2,n}$. Since G contains neither $K_{1,4}$, P_5, W_2^3, D_2, T_1^- nor $Z_{1,3}$, then $B_i \npreceq G$ for all $i \in \{1,2,3,4,5\}$ and $H_1 \npreceq G$. Then $k \geq 7$ and there is a graph $H_2 \in \mathcal{H}$ such that $H_2 \preceq G$. Since $H_2 \npreceq K_{1,3}$, $H_2 \npreceq Y_3$ and $H_2 \npreceq Z_{j,r}^-$ for $j \in \{1,2\}$ and all $r \geq 3$, then

- $H_2 = Z_{2,r_2}$ for some $r_2 \geq 3$.

By Lemma 12, there are graphs $H_1 \ldots H_{k-5}$ in $\mathcal{H}$ such that $H_i = Z_{i,r_i}$ for some $r_i \geq 3$ and all $1 \leq i \leq k-5$. From the same lemma, we have that $m \geq k-2$ and so $B_2 = P_{m+1}$ or $B_2 = Y_m$ for some $m \geq k-2$. Notice that $\{B_1, \ldots, B_5, H_1, \ldots, H_{k-5}\} \subseteq \mathcal{H}$. Since $|H| \leq k$, then $\mathcal{H}$ has no other graphs, namely, $\mathcal{H} = \{B_1, \ldots, B_5, H_1, \ldots, H_{k-5}\}$.

Consider $G = Z_{k-4,n}$. Since G contains neither $K_{1,4}$, P_{k-1}, W_2^3, D_2 nor T_1^- then $B_i \npreceq G$ for all $i \in \{1,2,3,4,5\}$. Furthermore, since $Z_{i,3} \npreceq G$ for all $1 \leq i \leq k-5$, then $H_i \npreceq G$ for all $1 \leq i \leq k-5$. Then G contains no graph of $\mathcal{H}$, which is a contradiction. $\square$

6 Applications

In this section we show an application of Theorem 3. In particular we show a family of forbidden subgraphs implying a hamilton path in large enough connected graphs.

Let N be the graph obtained by adding a pendant vertex to each vertex of a triangle. The graph N is often called "net". Consider the following Theorem.

Theorem 13 ([9]). *Let R and S be connected graphs. Then every $\{R,S\}$-free connected graph has a hamiltonian path if and only if $\{R,S\} \leq \{K_{1,3}, N\}$.*

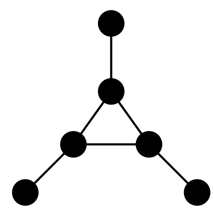

Figure 2: The graph N

We use now Theorem 3 to prove a variation of Theorem 13. We remove the limit on the number of forbidden subgraphs and replace it with the condition of the graph N being among the forbidden subgraphs.

Theorem 14. *Let $\mathcal{H}$ be a non-redundant family of connected graphs such that $N \in \mathcal{H}$. Then there is an integer $n \geq 1$ such that every $\mathcal{H}$-free connected graph G with $|V(G)| \geq n$ has a hamiltonian path if and only if $\mathcal{H} \in \mathbf{F}(3)$.*

Proof. Let $\mathcal{H} \in \mathbf{F}(3)$ with $N \in \mathcal{H}$. By Theorem 3, we know that every $\mathcal{H}$-free connected graph G with large enough order is $K_{1,3}$-free. Because $N \in \mathcal{H}$, then by Theorem 13, every $\mathcal{H}$-free connected graph G with large enough order has a hamiltonian path.

Let $\mathcal{H}$ be a family of connected graphs with $N \in \mathcal{H}$ and such that there is an n_0 with the property that every connected $\mathcal{H}$-free graph of order at least n_0 has a hamiltonian path. Let $n \geq \max(n_0, 4)$.

Consider the family $\mathcal{H}' = \mathcal{H}^3(n,n,n,n)$. All graphs in $\mathcal{H}'$ are connected and have order at least n_0. Moreover, none of them has a hamiltonian path. In the same way as in the proof of Theorem 3, we conclude that $\mathcal{H} \in \mathbf{F}(3)$. □

Theorem 14 is actually a case of implicit forbiddance, that we discussed in Section 1. Even though that, so far there was no result on forbidden subgraphs implying a hamilton path with families of large or infinite size and with an "if and only if" condition. So, we think the result is interesting by itself.

7 Conclusions

The characterization we were looking for is given by Theorem 3.

We have solved the problem of characterizing $\mathbf{H}(t)$ for any $t \geq 3$, but it is also possible to consider $t = 1$ and $t = 2$. It is not difficult to see that $\mathbf{F}(t)$ is also the solution for $t = 1$ and $t = 2$. In these cases, the corresponding families $\mathcal{H}^1(m,l,q,r)$ and $\mathcal{H}^2(m,l,q,r)$ after removing redundant graphs are as follows.

- $\mathcal{H}^1(m,l,q,r) = \{K_{1,l}, K_q, P_m\}$
- $\mathcal{H}^2(m,l,q,r) = \{K_{1,l}, W_q^2, Y_m^2, Z_{1,r}^2, \ldots, Z_{m-2,r}^2\}$

The case $t = 1$ is an easy proposition that can also be found in [5, Proposition 9.4.1].

Theorem 9 characterizes the families of forbidden subgraphs for claw-free graphs when restricting the size of the family. In the characterization there are four "irregular" families ($\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$ and $\mathbf{F}_6$) before the four infinite "regular" series ($\mathbf{F}_i^A$, $\mathbf{F}_i^B$, $\mathbf{F}_i^C$ and $\mathbf{F}_i^D$). We call them irregular because there is no easy way to see a pattern that describes them. They also include graphs that are claw-free, which are the result of the intersection of graphs that are not claw-free. These graphs become necessary because of the restriction in the size of the family. After $\mathbf{F}_6$, the families "stabilize" and appear the four infinite series.

It is also possible to consider restricting the size of the families of forbidden subgraphs for $K_{1,t}$-free graphs for $t \geq 4$. We think that a complete characterization of such families may be difficult and very long. In particular, we think that there might be many "irregular" families and many "regular" infinite series of families.

When searching for forbidden subgraphs implying some property $P(G)$ on graphs, it makes sense to study only forbidden subgraphs that imply $P(G)$ on graphs that satisfy some condition, usually related to the necessary conditions for satisfying $P(G)$. For example, in the case of graphs having a 2-factor, like in the Theorem 1, G should have minimum degree at least 2 and maximum degree at least 3. Minimum degree at least 2 is a necessary condition for having a 2-factor; maximum degree at least 3 is to avoid the trivial case a G being a cycle. Another example is the case of hamiltonian graphs, which have a necessary condition of being 2-connected, as studied for example in [11, 3].

Usual necessary conditions in the literature (hamilton cycle, hamilton-connected[2], 2-factor) appear to be connectivity and minimum degree conditions. When studying properties with such necessary conditions, Theorem 3 might not be useful to understand if a star is being implicitly forbidden or not. To try to find generalizations of Theorem 3 that can also be used in these cases, we propose the following two problems.

Problem 1. *Let $t \geq 3$ and $k \geq 1$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every large enough $\mathcal{H}$-free k-connected graph is $K_{1,t}$-free.*

In this paper we were able to resolve Problem 1 for the case $k = 1$.

Problem 2. *Let $t \geq 3$ and $d \geq 2$. Characterize all the families of connected graphs $\mathcal{H}$ satisfying the following property. Every $\mathcal{H}$-free large enough connected graph with minimum degree at least d is $K_{1,t}$-free.*

Even a combination of Problems 1 and 2 is possible.

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