# A spanning tree with high degree vertices

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**Abstract**

Let $G$ be a connected graph, let $X \subset V(G)$ and let $f$ be a mapping from $X$ to $\{2, 3, \dots\}$. In [3], Kaneko and Yoshimoto conjectured that if $|N_G(S) - X| \geq f(S) - 2|S| + \omega_G(S) + 1$ for any subset $S \subset X$, then there exists a spanning tree $T$ such that $d_T(x) \geq f(x)$ for all $x \in X$. In this paper, we show a result with a stronger assumption than this conjecture; if $|N_G(S) - X| \geq f(S) - 2|S| + \alpha(S) + 1$ for any subset $S \subset X$, then there exists a spanning tree $T$ such that $d_T(x) \geq f(x)$ for all $x \in X$.

## 1 Introduction

Let $G$ be a connected graph, let $X \subset V(G)$ and let $f$ be a mapping from $X$ to $\{2, 3, \dots\}$. For $x \in X$, we denote by $N_G(x)$ the neighborhood of $x$ in $G$. For $S \subset X$, we define $N_G(S) := \bigcup_{x \in S} N_G(x)$ and $f(S) := \sum_{x \in S} f(x)$. In this paper, we concentrate on a spanning tree $T$ such that $d_T(x) \geq f(x)$ for all $x \in X$. Frank and Gyárfás [2], and independently Kaneko and Yoshimoto [3] gave a necessary and sufficient condition for the existence of such a spanning tree, when $X$ is an independent set.

**Theorem 1 (Frank and Gyárfás [2], Kaneko and Yoshimoto [3])** *Let $G$ be a connected graph, let $X \subset V(G)$and let $f$ be a mapping from $X$ to $\{2, 3, \dots\}$. Suppose that $X$ is an independent set. Then there exists a spanning tree $T$ such that*

*$d_T(x) \geq f(x)$ for any $x \in X$ if and only if for any nonempty subset $S \subset X$,*

$$|N_G(S)| \geq f(S) - |S| + 1.$$

Now we consider the case where $X$ is not an independent set. In [3], Kaneko and Yoshimoto posed the following conjecture, and showed that the conjecture is true when $f(x) = 2$ for all $x \in X$. For $S \subset V(G)$, we denote by $\omega_G(S)$ the number of components of $G[S]$, where $G[S]$ denotes the subgraph of $G$ induced by $S$.

**Conjecture 2 (Kaneko and Yoshimoto [3])** *Let $G$ be a connected graph, and let $X \subset V(G)$. Let $f$ be a mapping from $X$ to $\{2, 3, \dots\}$. If for any nonempty subset $S \subset X$,*

$$|N_G(S) - X| \geq f(S) - 2|S| + \omega_G(S) + 1,$$

*then there exists a spanning tree $T$ such that $d_T(x) \geq f(x)$ for all $x \in X$.*

In this paper, we prove the following theorem. For $S \subset V(G)$, let $\alpha(S)$ be the independence number of $G[S]$.

**Theorem 3** *Let $G$ be a connected graph, and let $X \subset V(G)$. Let $f$ be a mapping from $X$ to $\{2, 3, \dots\}$. If for any nonempty subset $S \subset X$,*

$$|N_G(S) - X| \geq f(S) - 2|S| + \alpha(S) + 1, \tag{1}$$

*then there exists a spanning tree $T$ such that $d_T(x) \geq f(x)$ for all $x \in X$.*

This is weaker than Conjecture 2, but implies Theorem 1 as a corollary since $|N_G(S) - X| = |N_G(S)|$ and $\alpha(S) = |S|$ when $X$ is independent.

# 2 Lemmas

In this section, we prove two lemmas used in the proof of Theorem 3. Throughout this section, let $G$ be a connected graph, let $X \subset V(G)$ and let $f$ be a mapping from $X$ to $\{2, 3, \dots\}$. For a subgraph $H$ of $G$ and $x \in X$, we denote by $N_H(x)$ the neighborhood of $x$ in $H$, and we define $d_H(x) := |N_H(x)|$. For a subgraph $H$ of $G$ and $S \subset X$, let $d_H(S) := \sum_{x \in S} d_H(x)$. For $S \subset X$, let $b(S)$ be the number of components of $G[X]$ containing at least one vertex in $S$.

**Lemma 4** *If the condition (1) holds, then there exists a spanning forest $F$ of $G[X]$ such that for any nonempty subset $S \subset X$,*

$$|N_G(S) - X| \geq f(S) - d_F(S) - b(S) + 1. \tag{2}$$

An *independency tree* is a tree such that the set of its leaves is independent. Böhme, Broersma, Göbel, Kostochka and Stiebitz [1] characterized a connected graph having no spanning independency tree. The result implies the following theorem.

**Theorem 5 (Böhme et al. [1])** *Every connected graph $G$ has a spanning independency tree or a hamilton path.*

**Proof of Lemma 4.**

Let $C_1, C_2, \dots, C_p$ be components of $G[X]$. For each $C_i$, we can take a spanning independency tree or a hamilton path $F_i$, by Theorem 5. Let $F := \bigcup_{i=1}^{p} F_i$ and let $L$ be the set of leaves of $F$.

Now we fix $S \subset X$. Note that

$$d_F(S) \geq 2\big(|S| - |S \cap L|\big) + |S \cap L| = 2|S| - |S \cap L|.$$

For $1 \leq i \leq p$, we let $S_i := \{x_i\}$ if $F_i$ is a hamilton path of $C_i$ and both end vertices of $F_i$ are contained in $S$, where $x_i$ is one of end vertices of the hamilton path; otherwise let $S_i$ be the set of leaves of $F_i$ contained in $S$. Note that $S_i = \emptyset$ if $S \cap C_i = \emptyset$.

By the definition of an independency tree, $\bigcup_{i=1}^{p} S_i$ is an independent set, and hence $\alpha(S) \geq \sum_{i=1}^{p} |S_i|$. On the other hand, by the definition of $S_i$, $\sum_{i=1}^{p} |S_i| \geq |S \cap L| - b(S)$. Thus,

$$\alpha(S) \geq |S \cap L| - b(S).$$

By the condition (1) and the above two inequalities, we have

$$\begin{aligned} |N_G(S) - X| &\geq f(S) - 2|S| + \alpha(S) + 1 \\ &\geq f(S) - d_F(S) - b(S) + 1. \end{aligned}$$

Therefore the condition (2) holds. $\Box$

We call a graph isomorphic to $K_{1,m}$ a *star* if $m \geq 2$; the vertices with degree one are called *leaves* and the other vertex is called a *center*. We consider $K_{1,0}$ (and $K_{1,1}$) as a star with one center and no leaves (one leaf, respectively). Let $g$ be a mapping from $X$ to a set of nonnegative integers. Vertex-disjoint stars in $G$ are called a *$g$-star system for $X$* if the set of their centers is exactly $X$ and the stars with center $x$ have exactly $g(x)$ leaves for any $x \in X$. Using Hall's Marriage Theorem, we can obtain the following necessary and sufficient condition for the existence of a $g$-star system.

**Lemma 6** *Let $g$ be a mapping from $X$ to a set of nonnegative integers. Then $G$ has a $g$-star system for $X$ if and only if $|N_G(S) - X| \geq g(S)$ for any $S \subset X$, where $g(S) := \sum_{x \in S} g(x)$.*

**Proof of Lemma 6.**

First, we prove the necessary condition part. Suppose that $G$ has a $g$-star system for $X$. Then for any $S \subset X$, $|N_G(S) - X| \geq \sum_{x \in S} |L(x)| = g(S)$, where $L(x)$ is the set of leaves of a star with center $x$. Thus, the desired inequality holds.

Next we prove the sufficient condition part. Let $X := \{x_1, x_2, \dots x_{|X|}\}$, $I_0 := \{i : g(x_i) = 0\}$ and $I_1 := \{i : g(x_i) \geq 1\}$. For $i \in I_1$, we separate $x_i$ into $g(x_i)$ vertices $x_i^1, x_i^2, \dots, x_i^{g(x_i)}$. Let $\tilde{X} := \{x_j^i : j \in I_1, 1 \leq i \leq g(x_j)\} \cup \{x_j^0 : j \in I_0\}$. Consider the bipartite graph $H$ obtained from $G$ as follows; One partite set is $\tilde{X}$ and the other is $N_G(X) - X$, and we let $ux_j^i \in E(H)$ for $x_j^i \in \tilde{X}$ and for $u \in N_G(X) - X$ if and only if $u \in N_G(x_j)$. Then for any $\tilde{S} \subset \tilde{X}$, we obtain

$$|\tilde{S}| \leq g(S) \leq |N_G(S) - X| = |N_H(\tilde{S})|,$$

where $S = \{x_j \in X : x_j^i \in \tilde{S}\}$. By Hall's Marriage Theorem, we can find a matching of $H$ which covers $\tilde{X}$. Then we can construct a $g$-star system for $X$ in $G$ from this matching. $\square$

# 3 Proof of Theorem 3

Let $G$ be a connected graph satisfying the condition (1). By Lemma 4, there exists a spanning forest $F_X$ of $G[X]$ satisfying the condition (2). For $x \in X$, let

$$g(x) := f(x) - d_{F_X}(x) - 1$$

and let $X_g := \{x \in X : g(x) \geq 0\}$.

For any nonempty set $S \subset X_g$, we obtain $|N_G(S) - X| \geq f(S) - d_{F_X}(S) - b(S) + 1 > f(S) - d_{F_X}(S) - |S| = g(S)$ because $|S| \geq b(S)$. Then by Lemma 6, there exists a $g$-star system $F_g$ for $X_g$ in $G$ and let $F_0 := F_g \cup F_X$. Note that $F_0$ is a forest of $G$ and $d_{F_0}(x) \geq f(x)$ for any $x \in X - X_g$ and $d_{F_0}(x) = f(x) - 1$ for any $x \in X_g$.

For a forest $F$, we define $U(F) := \{x \in X_g : d_F(x) = f(x) - 1\}$. A forest $F$ is *$j$-saturated* if

(i) $E(F) \cap G[X] = E(F_X)$, so that $d_F(x) \geq f(x)$ for any $x \in X - X_g$,

(ii) $f(x) - 1 \leq d_F(x) \leq f(x)$ for any $x \in X_g$,

(iii) $N_F(v) - X = \emptyset$ for any $v \in V(G) - X_g$,

(iv) each component of $F$ has at least one vertex in $X$,

(v) $|U(F)| = |X_g| - j$.

Obviously $F_0$ is a 0-saturated forest. We will recursively construct a $j$-saturated forest $F_j$ from $j = 1$ to $j = |X_g|$. After finding a forest $F_{|X_g|}$, adding edges in order to make it connected, we can obtain the desired spanning tree.

Suppose that we obtain a $j$-saturated forest $F_j$ $(0 \le j \le |X_g| - 1)$. Note that $d_{F_j}(x) = f(x) - 1$ for $x \in U(F_j)$ and $d_{F_j}(x) = f(x)$ for $x \in X_g - U(F_j)$.

For a $j$-saturated forest $F$ and for $x \in X_g$, let $R_F(x) := N_F(x) - X$. For $X' \subset X_g$, similarly we define $R_F(X') := \bigcup_{x \in X'} R_F(x)$. For a $j$-saturated forest $F$ and for $X' \subset X_g$, $F$ is *$X'$-balanced* if $U(F) \subset X'$, $R_F(x) = R_{F_j}(x)$ for any $x \in X - X'$, and $R_F(X') = R_{F_j}(X')$. Note that if $X' \subset X'' \subset X_g$, then an $X'$-balanced forest is also $X''$-balanced. We take a subset $X_1 \subset X_g$ satisfying

(I) $U(F_j) \subset X_1 \subset \{x \in X_g : x \in U(F') \text{ for some } X_1\text{-balanced forest } F'\}$ and

(II) $\sum_{x \in X_1} \left|R_{F_j}(x)\right| - \left|R_{F_j}(X_1)\right| \ge b(X_1) - |U(F_j)|$.

Note that if we let $X_1 = U(F_j)$, then $X_1$ satisfies (I) and (II), because $F_j$ is $U(F_j)$-balanced and

$$\sum_{x \in U(F_j)} \left|R_{F_j}(x)\right| - \left|R_{F_j}\big(U(F_j)\big)\right| \ge 0 \ge b(U(F_j)) - |U(F_j)|.$$

Choose such $X_1$ so that $|X_1|$ is as large as possible. It follows from the condition (2) and the definition of $X_1$ that

$$\begin{aligned} |N_G(X_1) - X| &\ge f(X_1) - d_{F_X}(X_1) - b(X_1) + 1 \\ &= \sum_{x \in X_1} \left|R_{F_j}(x)\right| + |U(F_j)| - b(X_1) + 1 \\ &\ge \left|R_{F_j}(X_1)\right| + 1. \end{aligned}$$

Thus, there exists an edge $x_1y_1$ such that $x_1 \in X_1$ and $y_1 \in (N_G(X_1) - X) - R_{F_j}(X_1)$. By (I), there exists an $X_1$-balanced forest $F$ so that $d_F(x_1) = f(x_1) - 1$. If $y_1$ is not contained in $F$ or if $x_1$ and $y_1$ are contained in distinct components of $F$, $F \cup \{x_1y_1\}$ is a $(j+1)$-saturated forest. Therefore we may assume that $x_1$ and $y_1$ are contained in same component of $F$. Let $P$ be the unique path connecting $x_1$ and $y_1$ in $F$ and let $X_2 = \{x \in \big(X_g - X_1\big) \cap V(P) : R_F(x) \cap V(P) \ne \emptyset\}$. By (iii), there exists $x_2 \in X_g - X_1$ such that $y_1 \in N_P(x_2)$, and hence $X_2 \ne \emptyset$, Let $X_3 := X_1 \cup X_2$. Then $|X_3| > |X_1|$. Now we show that $X_3$ contradicts the maximality of $X_1$.

**Claim 1** *$U(F_j) \subset X_3 \subset \{x \in X_g : x \in U(F')$ for some $X_3$-balanced forest $F'\}$.*

*Proof.* Since $U(F_j) \subset X_1$ and $X_1 \subset X_3$, it follows that $U(F_j) \subset X_3$.

Let $x \in X_3$. It suffices to show that there exists an $X_3$-balanced forest $F'$ such that $x \in U(F')$. We first suppose that $x \in X_1$. Then by the definition of $X_1$, there exists an $X_1$-balanced forest $F'$ such that $x \in U(F')$. Since $X_1 \subset X_3 \subset X_g$, $F'$ is also an $X_3$-balanced forest. We next assume that $x \in X_2$. Notice that $d_F(x) = f(x)$ because $U(F) \subset X_1$. Since $R_F(x) \cap V(P) \neq \emptyset$, there exists a vertex $y$ in $R_F(x) \cap V(P)$. Let $F' = \big(F - xy\big) \cup \{x_1y_1\}$. Since $F'$ is a forest such that $d_{F'}(x_1) = d_F(x_1) + 1 = f(x_1)$ and $d_{F'}(x) = d_F(x) - 1 = f(x) - 1$, $F'$ is $j$-saturated and $x \in U(F')$. On the other hand, $R_{F'}(X_3) = R_F(X_3)$ by the definition of $F'$ and $R_F(X_3) = R_F(X_1) \cup R_F(X_2) = R_{F_j}(X_1) \cup R_{F_j}(X_2) = R_{F_j}(X_3)$ because $F$ is $X_1$-balanced. Hence $R_{F'}(X_3) = R_F(X_3) = R_{F_j}(X_3)$. Moreover, $U(F') \subset X_1 \cup \{x\} \subset X_3$ and $R_{F'}(z) = R_F(z) = R_{F_j}(z)$ for any $z \in X - X_3$. These imply that $F'$ is $X_3$-balanced. $\square$

Suppose that there exist $m$ components $C_1, C_2, \dots, C_m$ of $G[X]$ such that $V(C_i) \cap X_2 \neq \emptyset$ and $V(C_i) \cap X_1 = \emptyset$, where $m := b(X_3) - b(X_1)$. Note that $P$ passes through all component $C_i$ and $C_i \cap P$ is a path connecting vertices in $X_2$. Consider the direction of $P$ from $x_1$ to $y_1$. Let $z_i \in X_2$ be the first vertex in $C_i \cap P$ and let $z_i^-$ be the predecessor of $z_i$ along $P$ for $1 \leq i \leq m$. Let $Y_2 := \{y \in N_G(X) - X : |N_F(y) \cap X_2| = 2\}$. Then $z_i^- \neq z_j^-$ and $z_i^- \in Y_2 \cup \big(R_F(X_1) \cap R_F(X_2)\big)$ for $1 \leq i \neq j \leq m$. Therefore

$$
\begin{aligned}
&\sum_{x \in X_2} \big|R_F(x)\big| - \big|R_F(X_2) - R_F(X_1)\big| \\
&\quad= \sum_{x \in X_2} \big|R_F(x)\big| - \big|R_F(X_2)\big| + \big|R_F(X_2) \cap R_F(X_1)\big| \\
&\quad= |Y_2| + \big|R_F(X_2) \cap R_F(X_1)\big| \\
&\quad\geq \big|\{z_1^-, z_2^-, \dots, z_m^-\}\big| \\
&\quad= b(X_3) - b(X_1)
\end{aligned}
$$

On the other hand, since $F$ is an $X_1$-balanced forest, $R_{F_j}(x) = R_F(x)$ for any $x \in X_2$. Hence we obtain

$$
\begin{aligned}
&\sum_{x \in X_3} \big|R_{F_j}(x)\big| - \big|R_{F_j}(X_3)\big| \\
&\quad= \sum_{x \in X_1} \big|R_{F_j}(x)\big| - \big|R_{F_j}(X_1)\big| + \sum_{x \in X_2} \big|R_{F_j}(x)\big| - \big|R_{F_j}(X_2) - R_{F_j}(X_1)\big| \\
&\quad= \sum_{x \in X_1} \big|R_{F_j}(x)\big| - \big|R_{F_j}(X_1)\big| + \sum_{x \in X_2} \big|R_F(x)\big| - \big|R_F(X_2) - R_F(X_1)\big| \\
&\quad\geq \big(b(X_1) - |U(F_j)|\big) + \big(b(X_3) - b(X_1)\big) = b(X_3) - |U(F_j)|.
\end{aligned}
$$

This contradicts the maximality of $X_1$, and completes the proof of Theorem 3. $\square$

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