Spanning trees – A survey

Kenta Ozeki*
National Institute of Informatics,
Hitotsubashi 2-1-2, Chiyoda-ku,
Tokyo 101-8430, Japan
e-mail: ozeki@nii.ac.jp

Tomoki Yamashita
College of Liberal Arts and Sciences, Kitasato University
1-15-1, Kitasato, Sagamihara 228-8555, Japan
e-mail: tomoki@kitasato-u.ac.jp

July 18, 2010

Abstract

In this paper, we give a survey of spanning trees. We mainly deal with spanning trees having some particular properties concerning a hamiltonian properties, for example, spanning trees with bounded degree, with bounded number of leaves, or with bounded number of branch vertices. Moreover we also study spanning trees with some other properties, motivated from optimization aspects or application for some problems.

Keywords: Spanning trees, k-trees, k-ended trees, Branch vertices, k-leaf connected

1 Introduction

In this paper, a graph implies a finite simple graph, which has neither loops nor multiple edges. We give a survey of spanning trees. There are several problems on spanning trees which are generalizations of the hamiltonian path problem. A *hamiltonian path* of a graph is a path passing through all vertices of the graph. One of the huge targets of this problem is to find a necessary and sufficient condition for the existence of a hamiltonian path, except for a trivial one. However, it seems difficult and no one have succeeded.

It is well-known that the problem of determining whether a given graph has a hamiltonian path or not is NP-complete. For a graph G, by adding $k-2$ pendant

*Research Fellow of the Japan Society for the Promotion of Science edges to each vertex of G, we obtain a new graph G^*. Then it is clear that G^* has a spanning k-tree if and only if G has a hamiltonian path. Moreover, take one vertex v of G, and add $k-2$ pendant edges to v, and denote the resulting graph by G'. Then G' has a spanning tree with at most k leaves if and only if G has a hamiltonian path. Hence the following two problems are NP-complete: (i) a problem of determining whether a graph has a spanning k-tree or not and (ii) a problem of determining whether a graph has a spanning tree with at most k leaves or not. Therefore, it is widely believed that it is impossible to find a good necessary and sufficient condition for a graph to have a spanning k-tree or to have a spanning tree with at most k leaves. Thus we mainly deal with sufficient conditions for a graph to have such spanning trees. We begin with two famous results on hamiltonian paths.

Theorem 1 (Ore [149]) *Let G be a graph of order n. If $\sigma_2(G) \geq n-1$, then G has a hamiltonian path.*

Theorem 2 (Chvátal and Erdős [44]) *Let $m \geq 1$, and let G be an m-connected graph. If $\alpha(G) \leq m+1$, then G has a hamiltonian path.*

In Section 2–4, we discuss these generalizations to spanning trees with some properties concerning with hamiltonian paths.

A graph is *hamiltonian-connected* if every two vertices are connected by a hamiltonian path. A graph is *k-leaf-connected* if for all set S of k vertieces, there exists a spannning tree whose leaf set is precisely equal to S. Since a 2-leaf-connected graph is hamiltonian-connected, the concept of "k-leaf-connectedness" is a generalization of that of "hamiltonian-connectedness".

Theorem 3 (Ore [150]) *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n+1$, then G is hamiltonian-connected.*

In Section 5, we show some generalizations of this theorem for k-leaf-connectedness. In Section 6, we examine spanning trees studied from other motivation than hamiltonian properties. In Section 7, we research more than one spanning trees with some properties. In many applications, spanning trees of a multigraph are considered, however in this survey we mainly deal with spanning trees possesing the properties concering hamiltonian properties, and so we consider only simple graphs.

For standard graph-theoretic terminology not explained in this paper, we refer the reader to [178]. For a graph G and for a vertex v of G, the *degree of v*, denoted by $d_G(v)$, is the number of neighbors of v in G. Let $\alpha(G)$ and $\delta(G)$ be the independence number and the minimum degree of a graph G, respectively. If $\alpha(G) \geq k$, let $\sigma_k(G)$ be the minimum degree sum of an independent set of k vertices of G; otherwise we let $\sigma_k(G) = +\infty$. We denote the number of components of G by $\omega(G)$. For $X \subseteq V(G)$, the subgraph of G induced by X is denote by $G[X]$. A graph is said to be *$K_{1,t}$-free* if it contains no $K_{1,t}$ as an induced subgraph.

2 Spanning trees with bouded degree

In this section we consider spanning trees with bounded degree. For an integer $k \geq 2$, a *k-tree* is a tree with the maximum degree at most k. This topic mainly concerns with a connected factor, in particular, a connected $[1,k]$-factor. A *$[1,k]$-factor* is a spanning subgraph in which each vertex has the degree at least one and at most k. By the definition, a graph G has a spanning k-tree if and only if G has a connected $[1,k]$-factor. Because of this fact, the survey [122] on connected factors partly contains a topic on a spanning k-tree and a spanning f-tree. See also the survey [161] on a factor. This section is divided into three subsections, depending on the conditions, that is, toughness type condition, planarity condition, and independence number, connectivity and degree condition.

2.1 Toughness type condition

We first consider a necessary condition for a graph to have a hamiltonian cycle.

Proposition 4 *Let G be a graph having a hamiltonian cycle. Then for all $S \subseteq V(G)$, $\omega(G-S) \leq |S|$.*

However the converse does not hold. Therefore Chvàtal [42] introduced the concept of toughness. A graph G is *t-tough* if $\omega(G-S) \leq \frac{1}{t} \cdot |S|$ for all $S \subseteq V(G)$ with $\omega(G-S) \geq 2$, where t is a positive real number. The *toughness* of a graph G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough if G is not a complete graph; otherwise let $\tau(G) := +\infty$. Proposition 4 states that every graph having a hamiltonian cycle has the toughness at least one. Although the condition "the toughness at least one" cannot guarantee the existence of a hamiltonian cycle, higher toughness may be able to guarantee. Motivated by this fact, Chvàtal [42] conjectured that there exists a finite constant t_0 such that every t_0-tough graph has a hamiltonian cycle. This conjecture is still open even for a hamiltonian path, but Bauer, Broersma and Veldman [13] showed that $t_0 \geq \frac{9}{4}$ if such a constant t_0 exists. We refer for readers the survey on toughness and cycles [12].

We consider toughness type conditions for the existence of a spanning k-tree. The following proposition gives a necessary condition for a graph to have a spanning k-tree.

Proposition 5 *Let $k \geq 2$ and let G be a graph having a spanning k-tree. Then for all $S \subseteq V(G)$, $\omega(G-S) \leq (k-1)|S|+1$ and the equality holds if S is an independent set.*

Proof. Let T be a spanning k-tree of G. We show that $\omega(T-S) \leq (k-1)|S|+1$ by induction on $|S|$. When $|S| = 0$, there is nothing to prove. So we may assume that $|S| \geq 1$.

Take $v \in S$ and let $S' := S - \{v\}$. By induction hypothesis, $\omega(T - S') \leq (k-1)|S'| + 1$. Let C be a component of $T - S'$ containing v. Note that $d_C(v) \leq d_T(v) \leq k$. (When v is adjacent with no vertex of S' in T, then $d_C(v) = d_T(v)$.) Since the deletion of v divides C into $d_C(v)$ components, we obtain

$$\begin{aligned}\omega(T-S) &= \omega(T-S') - 1 + d_C(v) \\ &\leq (k-1)|S'| + 1 - 1 + k \\ &= (k-1)|S| + 1.\end{aligned}$$

Hence we have $\omega(T-S) \leq (k-1)|S|+1$. This implies that $\omega(G-S) \leq \omega(T-S) \leq (k-1)|S| + 1$. $\Box$

We call such a condition (containing a constant term) a toughness type condition. In [182], Win first considered a toughness type condition. Ellingham and Zha [66] gave a short proof to this theorem.

Theorem 6 (Win [182]) *Let $k \geq 2$ and let G be a connected graph. If $\omega(G-S) \leq (k-2)|S| + 2$ for all $S \subseteq V(G)$, then G has a spanning k-tree.*

Although we cannot increase the coefficient $(k-2)$ in Theorem 6 to larger than or equal to $(k-1)$ by Proposition 5, we do not know whether the coefficient $(k-2)$ is best possible. Theorem 6 implies that every $\frac{1}{k-2}$-tough graph has a spanning k-tree for $k \geq 3$. If we succeed to increase the coefficient $(k-2)$ in Theorem 6 to $(k-2+\varepsilon)$ for some constant $\varepsilon > 0$, then we obtain that every $\frac{1}{\varepsilon}$-tough graph has a spanning 2-tree.

As one of generalizations of Theorem 6, Ellingahm, Nam and Voss [65] gave a sufficient condition for the existence of a spanning f-tree. Let G be a connected graph and let f be a mapping from $V(G)$ to positive integers. Then a tree T of G is called *f-tree* if $d_T(x) \leq f(x)$ for all $x \in V(T)$. Note that when f takes a constant value k for every vertex, then an f-tree is equivalent to a k-tree, and hence f-tree is an extension of a concept of a k-tree.

Theorem 7 (Ellingham, Nam and Voss [65]) *Let G be a connected graph and let f be a positive integer-valued function on $V(G)$. If $\omega(G-S) \leq \sum_{v\in S}(f(v)-2)+2$ for all $S \subseteq V(G)$, then G has a spanning f-tree.*

Also Ellingahm et al. showed their result on toughness type condition implies some corollaries. One of them has been already shown by Zhenhong and Baoguang.

Theorem 8 (Zhenhong and Baoguang [193]) *Every m-edge-connected graph G has a spanning tree T such that $d_T(v) \leq \left\lceil \frac{d_G(v)}{m} \right\rceil + 2$ for all $v \in V(G)$.*

Enomoto, Ohnishi and Ota [68] gave another generalization of Theorem 6 in terms of total excess $\mathrm{te}(T,k)$, defined by $\mathrm{te}(T,k) = \sum_{x\in V(T)} \max\{d_T(x) - k, 0\}$ for a spanning tree T of a connected graph G. Later, Ohnishi and Ota [148] obtained a common generalization of them. For a spanning tree T of a connected graph G, and for a mapping f from $V(G)$ to positive integers, the total f-excess $\mathrm{te}(T, f)$ is defined by $\mathrm{te}(T, f) = \sum_{v \in V(T)} \max\{0, d_T(v) - f(v)\}$. Recently, Ozeki [152] obtained a further extension of this result.

Theorem 9 (Ohnishi and Ota [148]) *Let G be a connected graph and let f be a positive integer-valued function on $V(G)$, and let t be an integer. If $\omega(G - S) \leq \sum_{v \in S}(f(v) - 2) + 2 + t$ for all $S \subseteq V(G)$, then G has a spanning tree T with $\mathrm{te}(T, f) \leq t$.*

On the other hand, the similar proof of Proposition 5 implies a necessary condition for the existence of a spanning f-tree: if a graph G has a spanning f-tree, then $\omega(G - S) \leq \sum_{x \in S} \big(f(x) - 1\big) + 1$ for all $S \subseteq V(G)$. Frank and Gyáfás [82] showed that this necessary condition is also sufficient when $f(x) = +\infty$ for all $x \in V(G) - X$, where X is a specified independent set of G.

2.2 Spanning k-trees of graphs on surface

In this section, we consider spanning trees of graphs on surfaces. We also refer for the readers to the good survey [63].

Toughness type condition is useful in "Topological Graph Theory", since a graph on a surface has bounded toughness depending on the connectivity and the genus of the surface. For example, by using Theorem 6, Ellingham showed the following result. For a surface Σ, *Euler characteristic* χ is defined by $\chi = 2 - 2g$ if Σ is an orientable surface of genus g, and by $\chi = 2 - g$ if Σ is a nonorientable surface of genus g.

Theorem 10 (Ellingham [63]) *Let $m \geq 3$ be an integer and let G be an m-connected graph on a surface of Euler characteristic $\chi < 0$. Then G has a spanning $\lceil 2 + \frac{4-2\chi}{m^2-2m} \rceil$-tree.*

However, the value $\lceil 2 + \frac{4-2\chi}{m^2-2m} \rceil$ of Theorem 10 might not be tight, since Theorem 10 is proven by using Theorem 6 and we do not know that whether the toughness type condition of Theorem 6 is best possible for graphs with high connectivity. In fact, for the case $m = 3$ and $\chi \leq -36$, Sanders and Zhao [168] obtained the best value of k without using Theorem 6; every 3-connected graph on a surface of Euler characteristic $\chi \leq -36$ has a spanning $\lceil \frac{8-2\chi}{3} \rceil$-tree. Note that the complete bipartite graph $K_{3,6-2\chi}$ can be embedded in a closed surface of Euler characteristic χ and attains this upper bound of the maximum degree of a spanning tree. Recently, Ota and Ozeki [151] improved the upper bound on χ to $\chi \leq 0$.

Now we focus on a spanning 3-tree or 4-tree. For a graph G embedded on a closed surface F^2 which is not the plane, the *representativity* of G is the minimum number of intersecting points of G and γ, where γ ranges over all essential closed curves of F^2. Barnette [10] showed the following result on a spanning 3-tree.

Theorem 11 (Barnette [10]) *Every* 3*-connected planar graph has a spanning* 3*-tree.*

This result was extended to graphs on some surfaces having nonnegative Euler characteristic by Barnette [11]; every 3-connected graph on the projective plane, the torus, or the Klein bottle has a spanning 3-tree. Nakamoto, Oda and Ota [145] gave upper bounds of the number of vertices with degree 3 of the 3-tree; the upper bounds are $\frac{n-7}{3}$ and $\frac{n-3}{3}$ for a graph of order n on the plane or the projective plane, and on the torus or the Klein bottle, respectively. These upper bounds are best possible except for the plane case.

Next, we consider a graph on a surface of negative Euler characteristic with large representativity. Thomassen [174] showed that every triangulation of an orientable surface with large representativity has a spanning 4-tree. Moreover, he pointed out that there exists a triangulation of an orientable surface with arbitrary large representativity having no spanning 3-tree. Later, Yu [188] improved this result to general 3-connected graphs on a surface.

Theorem 12 (Yu [188]) *Every* 3*-conneted graph of an orientable surface with large representativity has a spanning* 4*-tree.*

Kawarabayashi, Nakamoto and Ota [117] extended this result and proved that every 3-connected graph on a surface of Euler characteristic $\chi \leq 1$ with large representativity has a spanning 4-tree T with at most $-2\chi - 1$ vertices of degree 4, that is, $\mathrm{te}(T, 3) \leq -2\chi - 1$. Ozeki [153] showed that every 3-connected graph on a surface of Euler characteristic $\chi \leq 0$ has a spanning $\lceil \frac{8-2\chi}{3} \rceil$-tree T with $\mathrm{te}(T, 3) \leq -2\chi - 1$ without the assumption of representativity.

As an analogue of Thomassen's result, Ellingham and Gao [64] showed that every 4-connected triangulation of an orientable surface with large representativity has a spanning 3-tree. Yu [188] improved this result as follows.

Theorem 13 (Yu [188]) *Every* 4*-connected graph on a surface with large representativity has a spanning* 3*-tree.*

Note that Archedeacon, Hartsfield and Little [5] constructed a k-connected triangulation of an orientable surface with representativity at least k which has no spanning $(k-1)$-tree. Therefore, we need the condition "large representativity". On the other hand, similarly as the case of spanning 4-trees in 3-connected graphs on surfaces, the following conjecture was posed;

Conjecture 14 *For each integer $\chi \leq 0$, there exists a constant c such that every* 4*-connected graph on a surface of Euler characteristic χ with large representativity* *has a spanning 3-tree with at most c vertices of degree 3.*

We can also consider other extensions of planar graphs. Note that every 3-connected planar graph has no $K_{3,3}$-minor. In this sense, Chen, Egawa, Kawarabayashi, Mohar and Ota [35] showed the existence of a spanning $(t+1)$-tree in a 3-connected graph which has no $K_{3,t}$-minor for $t \geq 3$. Recently, Ota and Ozeki improved their result as follows. Note that Chen et al. [35] also showed that the bound "$t-1$" of the maximum degree of a spanning tree is best possible.

Theorem 15 (Ota and Ozeki [151]) *Let $t \geq 4$ be an even integer. Let G be a 3-connected graph with no $K_{3,t}$-minor. Then G has a spanning $(t-1)$-tree.*

2.3 Independence number, connectivity and degree condition

Neumann-Lara and Rivera-Campo obtained an independence number and connectivity condition for a spanning k-tree. This result is a generalization of Theorem 2.

Theorem 16 (Neumann-Lara and Rivera-Campo [147]) *Let $m \geq 1$ and $k \geq 2$. Let G be an m-connected graph. If $\alpha(G) \leq m(k-1)+1$, then G has a spanning k-tree.*

In fact, they proved the following statement involving total excess. The case $k=3$ was obtained by Tsugaki using different method.

Theorem 17 (Neumann-Lara and Rivera-Campo [147], Tusgaki [175]) *Let $m \geq 1$ and $k \geq 3$ and $0 \leq c \leq m$. Let G be an m-connected graph. If $\alpha(G) \leq m(k-2)+c+1$, then G has a spanning k-tree T with $\mathrm{te}(T,k) \leq c$.*

Rivera-Campo [165] obtained an independence number and connectivity condition for a spanning k-tree containing a given matching.

On the other hand, Several authors gave degree sum conditions for a spanning k-tree. Win obtained a generalization of Theorem 1.

Theorem 18 (Win [180]) *Let $k \geq 2$ and let G be a connected graph of order n. If $\sigma_k \geq n-1$, then G has a spanning k-tree.*

Czygrinow, Fan, Hurlbert, Kierstead and Trotter [52] showed the same condition as Theorem 18 implies that (i) the graph has a spanning k-tree which is a caterpillar or (ii) G belongs a single exceptional class. Note that a *caterpillar* is a tree containing a path such that all other vertices have degree one.

Rivera-Campo [164] gave a result on degree sum condition, which implies Theorem 17 but does not imply Theorem 18. Fujisawa, Matsumura and Yamashita improved the result and obtained a common generalization of Theorem 17 and Theorem

18. For $\subseteq V(G)$ with $|S| \geq t$, let $\Delta_t(S) = \max\{\sum_{v\in T} d_G(v) \,|\, T \subseteq S, |T| = t\}$. We define $\sigma_t^s(G) := \min\{\Delta_k(S) \,|\, S$ is an independent set of G with $|S| = s\}$ if $\alpha(G) \geq s$; otherwise $\sigma_t^s(G) := +\infty$.

Theorem 19 (Fujisawa et al. [84]) *Let $m \geq 1$, $k \geq 3$, $c \geq 0$ and G be an m-connected graph of order n. If $\sigma_k^{m(k-1)+c+2}(G) \geq n - c - 1$, then G has a spanning tree T with maximum degree at most $k + \lceil c/m \rceil$ and $\mathrm{te}(T, k) \leq c$.*

Caro, Krasikov and Roditty [29] introduced a *k-frame*, and they gave a degree condition using k-frame for a graph to have a spanning k-tree. A independent set S of order k is a *k-frame* if $G - S'$ is connected for all $S' \subseteq S$. Kyaw [125] gave a neighborhood union condition using k-frame. This result improves results due to Caro et al. and Aung and Kyaw [7], and also is a generalization of a result of Flandrin, Jung and Li [80] for a hamiltonian path.

On the other hand, Matsuda and Matsumura gave an independence number condition for the existence of a spanning k-tree such that the set of the leaves contains some specified vertices.

Theorem 20 (Matsuda and Matsumura [139]) *Let k, s and m integers with $k \geq 2$, $0 \leq s \leq k$ and $m \geq s + 1$. Let G be an m-connected graph. Suppose that $\alpha(G) \leq (m-s)(k-1)+1$. Then for every $S \subseteq V(G)$ with $|S| = s$, G has a spanning k-tree T such that $S \subseteq L(T)$.*

However, such a spanning k-tree corresponds to a spanning f-tree with $f(x) = 1$ for specified vertices x and $f(x) = k$ for other vertices. Considering this fact, Enomoto and Ozeki posed the following conjecture for the existence of a spanning f-tree.

Conjecture 21 (Enomoto and Ozeki [69]) *Let $m \geq 1$ and let G be an m-connected graph and f be a mapping from $V(G)$ to positive integers. If $\sum_{x\in V(G)} f(x) \geq 2(|V(G)| - 1)$ and $\alpha(G) \leq \min\{\sum_{x\in R}(f(x) - 1) : R \subseteq V(G), |R| = m\} + 1$, then there exists a spanning f-tree.*

The condition "$\sum_{x\in V(G)} f(x) \geq 2(|V(G)| - 1)$" is a trivial necessary condition for the existence of a spanning f-tree. The independence number condition in Conjecture 21 is sharp if it is true. Enomoto and Ozeki [69] showed that Conjecture 21 holds when $s_1(f) + s_2(f) \leq m + 1$, where $s_i(f) := |\{x \in V(G) : f(x) = i\}|$. Note that this result is an improvement of Corollary 16 and Theorem 20.

Matsuda and Matsumura [139] also gave a degree sum condition for the existence of a spanning k-tree the set of whose leaves contains some specified vertices. By the same consideration as above, Enomoto posed the following problem.

Problem 22 (Enomoto [67]) *Find a sharp degree sum condition, together with some other conditions, for the existence of a spanning f-tree.*

Note again that for the existence of a spanning f-tree, we know that the condition "$\sum_{x\in V(G)} f(x) \geq 2(|V(G)|-1)$" is necessary. Moreover, some conditions on cut set may also be needed.

Here using a new notation $\mathrm{Cut}(G;f)$ instead of the connectivity, we introduce a partial solution of Problem 22. Let G be a connected graph and f be a mapping from $V(G)$ to positive integers. We define the connectivity with respect to f as follows; if G is not a complete graph,

$$\mathrm{Cut}(G;f) := \min\Big\{ \sum_{v\in S} \big(f(v)-1\big) : S \text{ is a cut set of } G \Big\},$$

otherwise let $\mathrm{Cut}(G;f)=0$. Using $\mathrm{Cut}(G;f)$, some results on a spanning k-tree or a spanning f-tree mentioned above can be improved. In particular, if we restrict ourselves to the case where $f(v)\geq 3$ for all $v\in V(G)$, the similar proofs directly imply results on a spanning f-tree. For example, a connected graph G has a spanning f-tree if $f(v)\geq 3$ for any $v\in V(G)$ and if $\alpha(G)\leq \mathrm{Cut}(G;f)+1$. This can be proven by the same method as the proof of Theorem 17, and this is an extension of the one due to Enomoto and Ozeki [69]. Also, Theorem 19 can be improved; a connected graph G has a spanning f-tree if $f(v)\geq 3$ for any $v\in V(G)$, and if $\sigma_k^r(G)\geq |G|-1$, where $r=\mathrm{Cut}(G;f)+2$ and $k=\min\{f(v): v\in V(G)\}$.

However, the difficulty of Problem 22 is based on the existence of vertices v with $f(v)=1$ or 2. In this sense, Problem 22 remains open for the general case, and this might be interesting.

At the end of this section, we consider a closure operation of Bondy-Chvátal type. Kano and Kishimoto [115] showed the following result; let G be an m-connected graph and let u and v be two non-adjacent vertices of G. Suppose that $d_G(u)+d_G(v)\geq |G|-m(k-2)-1$. Then G has a spanning k-tree if and only if $G+uv$ has a spanning k-tree. In addition, they also showed that the degree sum condition "$d_G(u)+d_G(v)\geq |G|-m(k-2)-1$" is best possible. Note that this theorem is an extension of the result on a closure for the hamiltonicity [18]. We point out that this theorem can be also improved for a spanning f-tree; let G be a connected graph, let f be a mapping from $V(G)$ to positive integers, and let u and v be two non-adjacent vertices of G. Suppose that $d_G(u)+d_G(v)\geq |G|-\mathrm{Cut}'(G;f)-1$, where $\mathrm{Cut}'(G;f) := \min\big\{\sum_{x\in S}\big(f(x)-2\big) : S \text{ is a cut set of } G\big\}$. Then G has a spanning f-tree if and only if $G+uv$ has a spanning f-tree.

3 Spanning k-ended tree

A tree with at most k leaves is called a k-ended tree. Win [181] obtained a generalization of Theorem 2, which was conjectured by Las Vergnas. Broersma and Tuinstra [26] gave a generalization of Theorem 1.

Theorem 23 (Win [181]) *Let $k \geq 2$ and let G be an m-connected graph. If $\alpha(G) \leq m + k - 1$, then G has a spanning k-ended tree.*

Theorem 24 (Broersma and Tuinstra [26]) *Let $k \geq 2$ and let G be a connected graph of order $n \geq 2$. If $\sigma_2(G) \geq n - k + 1$, then G has a spanning k-ended tree.*

Tsugaki and Yamashita obtained a common generalization of these theorem in terms of σ_t^s.

Theorem 25 (Tsugaki and Yamashita [176]) *Let $m \geq 1$ and $k \geq 2$, and let G be an m-connected graph of order n. If $\sigma_2^{m+k}(G) \geq n-k+1$, then G has a spanning k-ended tree.*

On the other hand, Flandrin, Kaiser, Kužel, Li and Ryjáček [81] obtained a neighborhood union condition for spanning k-ended trees. This is a generalization of a result due to Bauer, Fan and Veldman [14]. We denote by $N_k(G)$ the minimum order of the neighborhoods of an independent set of order k.

Theorem 26 (Flandrin et al. [81]) *Let $k \geq 2$ and let G be a connected graph of order n. If $N_k(G) > \frac{k}{k+1}(n - k)$, then G has a spanning k-ended tree.*

Here we also consider a closure operation of Bondy-Chvátal type for a spanning k-ended tree. Broersma and Tuinstra [26] showed that for a pair of two vertices u and v in a graph G of order n with $d_G(u)+d_G(v) \geq n-1$, G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree. Although they also showed that the degree condition "$d_G(u) + d_G(v) \geq n - 1$" is best possible, Fujisawa, Saito and Schiermeyer [85] improved this result defining the notation of "the distant area".

We now consider claw-free graphs and $K_{1,4}$-free graphs. A $K_{1,3}$-free graph is called *claw-free*. Matthews and Sumner [141] obtained a degree sum condition for a claw-free graph to have a hamiltonian path. Kano, Kyaw, Matsuda, Ozeki, Saito and Yamashita obtained a slightly stronger result than a generalization of it.

Theorem 27 (Kano et al. [116]) *Let $k \geq 2$ and let G be a connected claw-free graph of order n. If $\sigma_{k+1}(G) \geq n - k$, then G has a spanning k-ended 3-tree.*

Recently, Kyaw showed a result for $K_{1,4}$-free graphs:

Theorem 28 (Kyaw [127]) *Let G be a connected $K_{1,4}$-free graph of order n. (i) If $\sigma_3(G) \geq n$, then G has a hamiltonian path. (ii) If $\sigma_{k+1}(G) \geq n - k/2$ ($k \geq 3$), then G has a spanning k-ended tree.*

Lastly we introduce some results from point of view of optimization. The problem of finding a spanning tree with smallest number of leaves, called MinLst (Minimum Leaf Spanning Tree), is NP-hard because it is an generalization of the hamiltonian path problem. Lu and Ravi [137] showed that there is no constant factor approximation algorithm for it, unless $P = NP$. Therefore Salamon and Wiener [167] considered the equivalent problem, the problem of finding a spanning tree with maximum number of non-leaf vertices, called MaxIST (Maximum Internal Spanning Tree), and they gave a 2-approximation algorithm for MaxIST, that is, an algorithm to find a spanning tree such that its number of non-leaf vertices is at least half of the number of non-leaf vertices of a spanning tree with maximum non-leaf vertices. Salamon [166] improved it to 7/4-approximation. Salamon and Wiener [167] also gave a 3/2- and a 6/5-approximation for claw-free graphs and cubic graphs, respectively.

4 Spanning trees with few branch vertices

We present results on trees with few branch vertices. A *branch vertex* of a graph is a vertex of degree greater than two. One of the interest in the existence of spanning trees with bounded branch vertices arises in the realm of multicasting in optical networks; see [88, 89, 90].

Gargano, Hammar, Hell, Stacho and Vaccaro posed the following conjecture, and proved the case $k = 1$.

Conjecture 29 (Gargano et al. [89]) *Let k be a non-negative integer. Let G be a connected graph of order n. If $\sigma_{k+2}(G) \geq n - 1$, then G has a spanning tree with at most k branch vertices.*

This degree sum condition seems to be not best possible if it is true. Hence, we propose a conjecture involving a sharp degree sum condition. Flandrin et al. [81] also pointed out that $\sigma_{k+2}(G)$ may be replaced with $\sigma_{k+3}(G)$.

Conjecture 30 *Let k be a positive integer and let G be a connected graph of order n. If $\sigma_{k+3}(G) \geq n - k$, then G has a spanning tree with at most k branch vertices.*

Also, in [89], Gargano et al. obtained degree sum conditions for claw-free graphs and $K_{1,4}$-free graphs to have a spanning tree with at most k branch vertices. These degree sum conditions seem to be strong. In fact, these results are implied from Theorem 27 and Theorem 28. Matsuda, Ozeki and Yamashita [140] proved that a connected claw-free graph G has a spanning tree with at most k branch vertices if $\alpha(G) \leq 2k + 2$. They also showed that the condition is best possible. This result suggests that we should consider a σ_{2k+3} condition for a spanning tree with at most k branch vertices. In the same paper, they made the following conjecture, and showed that the case $k = 1$ of this conjecture holds.

Conjecture 31 (Matsuda et al. [140]) *Let G be a connected claw-free graph of order n. If $\sigma_{2k+3}(G) \geq n-2$, then then G has a spanning tree with at most k branch vertices.*

A *spider* is a tree with at most one branch vertex. A branch vertex of a spider is called the *center* of the spider. If a spider is a path, then every vertex can be viewed as the center. Gargano and Hammer [88] gave degree conditions for a bipartite graph to have a spanning spider or a spanning spider with prescribed center. Flandrin et al. [81] obtained degree sum conditions for a spanning spider with prescribed center.

5 k-leaf connected graphs

A graph G is said to be *k-leaf-connected* if $|V(G)| > k$ and for each subset S of $V(G)$ with $|S| = k$, G has a spanning tree T with $L(T) = S$, where $L(T)$ is the set of leaves of T. Recall that the concept of "k-leaf-connectedness" is a generalization of that of "hamiltonian-connectedness", since a 2-leaf-connected graph is hamiltonian-connected. Gurgel and Wakabayashi [96] gave an Ore-type condition for a graph to be k-leaf-connected; for a graph G of order n, if $\sigma_2(G) \geq n + k - 1$, then G is k-leaf-connected. This result is derived from the assertion that the property of being k-leaf-connected is stable under a closure operation of Bondy-Chvátal type. Later, Egawa, Matsuda, Yamashita and Yoshimoto improved this result as follows.

Theorem 32 (Egawa et al. [59]) *Let $k \geq 2$ be an integer. Let G be a $(k+1)$-connected graph of order n. If $\sigma_2(G) \geq n + 1$, then G is k-leaf-connected.*

Since the condition "$\sigma_2 \geq |V(G)| + k - 1$" implies that the connectivity is at least $k + 1$, Theorem 32 is actually an improvement of the result due to Gurgel and Wakabayashi.

The above results concern degree sum conditions, and there is no result on independence number and connectivity condition. Motivated by this fact, it may be interesting to consider a generalization of the following theorem for the direction of k-leaf-connectedness.

Theorem 33 (Chvátal and Erdős [44]) *Let G be an m-connected graph. If $\alpha(G) \leq m - 1$, then G is hamiltonian-connected.*

6 Other properties

In this section, we discuss spanning trees with some properties which are not generalizations of a hamiltonian path. However, we study such spanning trees from other aspects, in particular, optimization.

6.1 Spanning tree with many leaves

In 1981, Storer announced (without proof) that every connected cubic graph with n vertices has a spanning tree with at least $\frac{1}{4}n + 2$ leaves. Linial conjectured that every connected graph with n vertices and with minimum degree k has a spanning tree with at least $\frac{k-2}{k+1}n + c_k$ leaves, where c_k is a constant depending only on k. Caro, West and Yuster [30] pointed out that Alon's result disproves this conjecture for sufficiently large k. Alon [1] (See also [4]) proved by probabilistic method that for large n there exists a graph with the minimum degree at least k and with no dominating set of size less than $(1+o(1))\frac{1+\ln(k+1)}{k+1}n$. For a graph G, a subset S of $V(G)$ is a *dominating set* if every vertex of G is in S or is adjacent to a vertex in S, and moreover if $G[S]$ is connected then S is a *connected dominating set*. Note that a vertex subset is a connected dominating set if and only if its complement is contained in the set of leaves of some spanning tree. Therefore there exists a graph with the minimum degree at least k and with no spanning tree having more than $(1+o(1))\frac{k-\ln(k+1)}{k+1}n$ leaves. However, for small values of k, it is known that Linial's conjecture is true. Linial and Sturtevant (Unpublished) and Kleitman and West [121], independently, proved it for $k = 3$ with $c_3 = 2$. Griggs and Wu [93] proved Linial's conjecture for $k = 4, 5$ with $c_4 = \frac{8}{5}$ and $c_5 = 2$. The best bounds for fixed k greater than five remain open.

We consider graphs with certain forbidden induced subgraphs. Griggs, Kleitman and Shastri [92] showed that every connected cubic graph with n vertices that contains no induced diamond has a spanning tree with at least $\frac{1}{3}n + \frac{4}{3}$ leaves, where a *diamond* is the graph K_4 minus one edge. Bonsma [19], by defining a cubic diamond, showed every connected graph with n vertices with $\delta \geq 3$ that contains no cubic diamond has a spanning tree with at least $\frac{2}{7}n + \frac{12}{7}$ leaves. Furthermore, Bonsma [19] proved that every connected triangle-free graph with n vertices with $\delta \geq 3$ has a spanning tree with at least $\frac{1}{3}n + \frac{4}{3}$ leaves. Bonsma and Zickfeld [20] defined a 2-necklace and a 2-blossom, and proved that every connected graph with n vertices with $\delta \geq 3$ without 2-necklaces or 2-blossoms has a spanning tree with at least $\frac{1}{3}n + \frac{4}{3}$ leaves.

We consider the number of edges for a spanning tree with many leaves. For any integer $n > t \geq 2$, Ding, Johnson and Seymour [58] determined the smallest $f(n,t)$ such that every connected graph with n vertices and at least $f(n,t)$ edges must have a spanning tree with more than t leaves.

Now we show some results on spanning trees with many leaves from the aspect of optimization. The problem of finding a spanning tree with maximum leaves in a given graph is known as NP-hard [87], even for cubic graphs [129]. Thus, some approximation algorithms for this problem have been considered. Lu and Ravi [138] gave a 3-approximation algorithm, that is, an algorithm to find a spanning tree such that the number of leaves of it is at least one third of that of the spanning tree with maximum leaves. Later, Solis-Oba [171] improved this and gave a 2-approximation algorithm. Approximation algorithms for cubic graphs are known [21, 45, 136].

This problem for the bipartite graph version has been also considered. Precisely, some researches are studying the problem of finding a spanning tree of a given bipartite graph with one partite set X such that the number of leaves contained in X is maximum. This problem was first posed by Rahman and Kaykobad [163], and Li and Toulpise [133] showed that the problem of finding it is NP-hard. Later Fusco and Monti [86] gave an approximation algorithm for regular bipartite graphs. They also showed that for a cubic bipartite graph G with one partite set X, G has a spanning tree with at least $\frac{|X|}{3} + 1$ leaves contained in X.

Alon, Fomin, Gutin, Krivelevich and Saurabh [2, **?**] considered a directed version of this problem.

6.2 Locally connected spanning trees

A spanning tree T of a connected graph G is called *locally connected spanning tree* if for any vertex $v \in V(G)$, the neighborhood of v in T is connected in G, that is, $G[N_T(v)]$ is connected. This notion was first introduced by Cai in [27].

In [28], it is shown that the problem of determining whether a given graph contains a locally connected spanning tree is NP-complete, even if we restrict ourselves to planar graphs or split graphs. A graph G is called a *split graph* if there exists a partition $V \cup I$ of $V(G)$ such that V is a clique and I is an independent set. Since split graph is also chordal graph, locally spanning tree problem is NP-complete even for chordal graphs. A graph is *chordal graph* if every cycle of length at least four has a chord. However, Lin, Chang and Chen [134] gave a linear-time algorithm for finding a locally connected spanning tree in strongly chordal graphs. A chordal graph G is *strongly* if G has no induced subgraph H which has a hamiltonian cycle $x_1y_1x_2y_2 \ldots x_ny_nx_1$ ($n \geq 3$) such that $d_H(x_i) = 2$.

6.3 Degree-preserving spanning trees

In [130], Lewinter introduced a notion "degree-preserving". For a connected graph G and a spanning tree T of G, a vertex $v \in V(G)$ is called *degree-preserving in T* if $d_T(v) = d_G(v)$. Lewinter [130] showed that if G has spanning trees with exactly k and l degree-preserving vertices, respectively, then G also has a spanning tree with exactly p degree-preserving vertices for each p, $l \leq p \leq k$. (This result was proven using the adjacency tree graph, see Section 7.4.) Choi and Guan [41] determined the maximum number of degree-preserving vertices in a spanning tree of a hypercube.

Broersma, Koppius, Tuinstra, Huck, Kloks, Kratsch and Müller [24] defined *degree-preserving spanning tree* of a graph G by the meaning a spanning tree of G such that as many vertices of T as possible is degree-preserving. They also showed that the problem of finding a degree preserving spanning tree of a given graph is NP-hard even when restricted to split graphs or bipartite planar graphs with maximum degree at most 6. Damaschke [56] showed that the problem is also NP-hard even if we deduce the maximum degree to 5 for bipartite planar graphs and to 3 for general planar graphs.

6.4 Average distance

For a connected graph G of order n, we define the *average distance* $\mu(G)$ *of* G, as $\mu(G) := \binom{n}{2}^{-1} \sum_{x,y \in V(G)} \mathrm{dist}_G(x,y)$. We refer surveys for the average distance [70, 160]. In this subsection, we consider spanning trees with small average distance.

Johnson, Lenstra and Rinnooy-Kan [109] proved that the problem of finding a spanning tree of a given graph with minimum average distance among all spanning trees is NP-hard. Several researchers have considered approximation algorithms for finding it [183, 184, 185] and Fischetti, Lancia and Serafinl [79] gave exact algorithms. On the other hand, when we restrict ourselves to some particular classes, it is known that we can find a minimum average distance spanning tree in polynomial-time, for example, distance-hereditary graphs by Dahlhaus, Dankelmann, Goddard and Swart [53], and interval graphs by Dahlhaus, Dankelmann and Ravi [54].

We consider the upper bound of $\mu(T)$ of spanning trees T in G. Entringer, Kleitman and Szekely [71] showed that a connected graph G has a spanning tree T with $\mu(T) \leq 2\mu(G)$. Dankelmann and Entringer [55] showed that a connected graph G has a spanning tree T with $\mu(T) \leq |V(G)|/(\delta(G)+1)+5$.

6.5 Leaf degree and leaf distance

In [111], Kaneko posed two new concepts concerning a spanning tree, called *leaf degree* and *leaf distance*. Let G be a connected graph and let T be its spanning tree. For a vertex v of $V(G)$, the *leaf degree of* v in T is defined as the number of leaves of T adjacent to v. Kaneko gave a necessary and sufficient condition for a graph to have a spanning tree with bounded leaf degree. We denote by $i(G)$ the number of isolated vertices of a graph G.

Theorem 34 (Kaneko [111]) *Let G be a connected graph and let m be a positive integer. Then G has a spanning tree with maximum leaf degree at most m if and only if $i(G-S) < (m+1)|S|$ for every nonempty subset $S \subseteq V(G)$.*

Recently, Szabó [172] improved Theorem 34 and showed a result on the order of a largest tree with bounded leaf degree.

For a tree T, the *leaf distance of* T is defined the minimum distances in T between any two leaves of T. Kaneko proposed the following conjecture.

Conjecture 35 (Kaneko [111]) *Let d be an integer with $d \geq 3$ and let G be a connected graph of order at least $d+1$. If $i(G-S) < \dfrac{2|S|}{d-2}$ for any nonempty subset $S \subseteq V(G)$, then G has a spanning tree with leaf distance at least d.*

Note that a tree with leaf distance at least 3 is equivalent to a tree with maximum leaf degree at most 1. Therefore the case $d = 3$ of Conjecture 35 is obtained from the case $m = 1$ of Theorem 34. Recently, Kaneko, Kano and Suzuki [112] solved the case $d = 4$. The case $d \geq 5$ is still open.

6.6 Spanning trees with high degrees

Let G be a connected graph and let X be a subset of $V(G)$. Let g be a mapping from X to the set of integers. In this subsection, we focus on the existence of a spanning tree T in G such that $d_T(x) \geq g(x)$ for all $x \in X$. Note that we obtain the following necessary condition for the existence of such a spanning tree. For a graph G and $S \subseteq V(G)$, let $\omega_G(S)$ be the number of conponents of $G[S]$.

Proposition 36 *Let G be a connected graph, let $X \subseteq V(G)$ and let g be a mapping from X to the set of integers. If there exists a spanning tree T in G such that $d_T(x) \geq g(x)$ for any $x \in X$, then for any nonempty subset $S \subseteq X$,*

$$\Big| \bigcup_{x \in S} N_G(x) - S \Big| - g(S) + 2|S| - \omega_G(S) \geq 1.$$

Frank and Gyárfás [82], and independently, Kaneko and Yoshimoto [114] proved that this necessary condition is also sufficient when X is an independent set. Later, Egawa and Ozeki extended this result as follows:

Theorem 37 (Egawa and Ozeki [60]) *Let G be a connected graph and let $X \subseteq V(G)$, and let g be a mapping from X to the set of integers. Suppose that $G[X]$ has no induced path of order four. Then there exists a spanning tree T such that $d_T(x) \geq g(x)$ for any $x \in X$ if and only if for any nonempty subset $S \subseteq X$,*

$$\Big| \bigcup_{x \in S} N_G(x) - S \Big| - g(S) + 2|S| - \omega_G(S) \geq 1.$$

Note that when $G[X]$ has an induced path of order four, there exist infinitely many graphs satisfying the necessary condition but having no desired spanning tree. In this sense, the condition "$G[X]$ has no induced path of order four" cannot be weakened.

6.7 Independency spanning trees

When we look for a spanning tree of a connected graph by depth first search, we obtain a spanning tree whose leaves are pairwise nonadjacent, called *independency tree,* unless it is a hamiltonian path with adjacent end vertices. Böhme, Broersma, Göbel, Kostochka and Stiebitz [17] showed that a connected graph has no independency trees if and only if it is a cycle, a complete graph or a balanced complete bipartite graph.

6.8 Spanning trees which are isomorphic to particular trees

We consider sufficient conditions for graphs to have a subgraph which is isomorphic to some particular tree or forest.

In 1959, Erdős and Gallai [74] proved that if a graph G of order n satisfies $|E(G)| > (k-1)n/2$, then G has a path of length k. Motivated by this result, in 1963, Erdős and Sós posed the following conjecture.

Conjecture 38 (Erdős-Sós) *Let G be a graph of order n. If $|E(G)| > (k-1)n/2$, then G has every tree of k edges.*

A *leg* of a spider is a path from the center to a vertex of degree one. Recall that a spider is a tree with at most one branch vertex. Fan and Sun [76] showed that Erdős-Sós conjecture is true for spiders with three legs and also for spiders that has no leg of length more than four.

In [77], Faudree, Rousseau, Schelp and Schuster defined a concept "panarboreal" and gave a degree condition for a graph to be panarboreal. A connected graph G of order n is said to be *panarboreal* if for every tree T of order n, G has a spanning tree which is isomorphic to T. In other words, G contains all trees of order n as spanning trees.

Theorem 39 (Faudree et al. [77]) *Let $k \geq 3$ and let $n \geq 3k^2 - 9k + 8$. For a graph G of order n, if $\Delta(G) = n - 1$ and $\delta(G) \geq n - k$, then G is panarboreal.*

Clearly, any panarboreal graph of order n must have a vertex of degree $n-1$ in order to have the star $K_{1,n-1}$, and hence the condition $\Delta(G) = n - 1$ is necessary. However, if we restrict ourselves to some particular class of trees, we may not need the maximum degree condition. Erdős, Faudree, Rousseau and Schelp gave the following result.

Theorem 40 (Erdős et al. [73]) *Let $k \geq 2$ and let $n \geq 2(3k-2)(2k-3)(k-2)+1$. If G is a graph of order n with $\delta(G) \geq n - k$, then G contains every tree T with $\Delta(T) \leq n - 2k + 2$ as a spanning tree.*

On the other hand, some researchers have tried to find not only a spanning tree but also a subtree of a graph. The most basic result is the following.

Theorem 41 (Chvàtal [43]) *For every tree T with k edges, if G is a graph with $\delta(G) \geq k$, then G has a subgraph which is isomorphic to T.*

Brandt [22] improved the above result for all forests and Babu and Diwan [9] improved the condition "$\delta(G) \geq k$" to "$\sigma_2(G) \geq 2k+1$". Zioło [194, 195] considered the bipartite digraph analogue of the above results and gave the minimum indegree and outdegree condition for bipartite digraphs to have any directed trees or forests as subgraphs.

In [22], Brandt gave an edge number condition on the existence of subforest, which was conjectured by Erdős and Sós in [72].

Theorem 42 (Brandt [22]) *Let G be a graph with n vertices. If*

$$|E(G)| \geq \max\left\{ \binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1) \right\},$$

then G has every forest with k edges and without isolated vertices as a subgraph.

However, when we want to show that a given graph G is panarboreal, is it necessary to consider all trees of order $|V(G)|$? For example, if G has both a path and a star as spanning trees, G must have a spanning tree which is isomorphic to a tree obtained from the star of order $n-1$ by subdividing one of its edges. Bridgland, Jamison and Zito [23] defined a *spanning-tree forcing set*. A set S of trees of order n *forces* a tree T if every graph having each tree in S as a spanning tree must also have T as a spanning tree. A *spanning-tree forcing set* is the set of trees that forces every trees of order n. They showed that the star belongs to every spanning-tree forcing set for order $n > 1$, and the path belong to every spanning-tree forcing set for order $n \notin \{1, 6, 7, 8\}$.

7 More than one spanning tree

7.1 Matrix-Tree Theorem

The most classical interest concerning with a spanning tree is the number of spanning trees of a given graph. Kirchhoff [120] gave a formula for determining it, which is known as the *Matrix-Tree Theorem*; The number of spanning trees of a graph G is the value of any cofactor of the matrix $D(G) - A(G)$, where $D(G)$ is the degree matrix (the ith diagonal entry is equal to the degree of ith vertex and the other entry is equal to zero) and $A(G)$ is the adjacency matrix of G (the entry (i, j) is equal to the number of edges between ith vertex and jth vertex), respectively. This topic is still much studied, in particular, explicit formulas (or bounds) of the number of spanning trees for some special classes. That for complete graphs is most famous among such classes; The number of spanning trees of K_n is n^{n-2}, called *Cayley's Formula* [34]. Several proofs of Cayley's Formula are known, and the most famous one is due to Prüfer [162]. See the good book [144] by Moon for Cayley's Formula and the Matrix-Tree Theorem.

The explicit formulas of the number of spanning trees are known for other classes than complete graphs; complete multipartite graphs [8, 62, 61, 131], regular graphs [51, pp.95], and circulant graphs [6, 38, 186, 192], and so on. Some upper bounds of the number of spanning trees are also studied, see [57, 78, 94, 95]. The survey on Laplacian matrices [142] also contains the topic of the Matrix-Tree Theorem.

On the other hand, a graph is called *t-optimal for n vertices and m edges* if it has the maximum number of spanning trees among all graphs with n vertices and m edges. Some researchers have been considered a *t-optimal graph* for given integers n and m, see [16, 39, 91, 118, 156, 158].

7.2 Edge-disjoint spanning trees

In this section, we concentrate on the existence of edge-disjoint spanning trees. The results in this section hold for multigraphs. The most famous and basic result is obtained by Nash-Williams and Tutte, independently. In addition, we obtain the following corollary.

Theorem 43 (Nash-Williams [146], Tutte [177]) *Let G be a connected graph. Then G has k edge-disjoint spanning trees if and only if $|F| \geq k\,(\omega(G-F)-1)$ for any $F \subseteq E(G)$.*

Corollary 44 *Let G be a $2k$-edge-connected graph. Then G has k edge-disjoint spanning trees.*

Catlin imporved Corollary 44 and obtained a necessary and sufficient condition for the existence of k edge-disjoint spanning trees using the term "edge-connectivity".

Theorem 45 (Catlin [31]) *Let G be a connected graph. Then G is $2k$-edge-connected, ($(2k+1)$-edge-connected) if and only if $G-F$ has k ($k+1$, respectively) edge-disjoint spanning trees for any $F \subseteq E(G)$ with $|F| = k$.*

As a corollary of Theorem 43, we can calculate the maximum number of edge-disjoint spanning trees of a given graph. In [155], Palmer determined the number for several families of graphs; quasi-random graphs, regular graphs, complete bipartite graphs, cartesian products and the hypercubes and so on.

Cunningham [47] obtained a necessary and sufficient condition for a graph G to have p spanning trees such that each edge of G lies in at most q spanning trees of these p spanning trees. Catlin, Grossman, Hobbs and Lai [32] extended this result to matroids. As an extension of Theorem 43, Chen, Koh and Peng [36] obtained a necessary and sufficient condition for the existence of spanning forests with a prescribed number of components. In [37], Chen and Lai gave a short proof of it. Recently, Catlin, Lai and Shao [33] obtained a generalization of Theorem 45 as like this result.

On the other hand, several researchers have considered the existence of k edge-disjoint trees all of which contains specified vertices. Let G be a graph and let $S \subseteq V(G)$. An *S-Steiner tree* is a tree containing all vertices of S. A subset S of $V(G)$ is called *k-edge-connected (in G)* if for every pair of two vertices u, v in S, there exist at least k edge-disjoint paths connecting u and v. Kriesell posed the following conjecture on the existence of k edge-disjoint S-Steiner trees, which is an extension of Theorem 43. This edge-connectivity condition is best possible.

Conjecture 46 (Kriesell [123]) *Let G be a graph and let $S \subseteq V(G)$. If S is $2k$-edge-connected, then G has k edge-disjoint S-Steiner trees.*

Let $f_l(k)$ ($g_l(k)$) be a function such that for every $f_l(k)$-edge-connected vertex set S of a graph with $|S| \leq l$ ($|V(G) - S| \leq l$) there exist k edge-disjoint S-Steiner trees. Petingi and Rodriguez [157] showed $g_l(k) = 2k\left(\frac{3}{2}\right)^l$. Kriesell [123] proved that Conjecture 46 is true if $G - S$ is Eulerian, and this implies $g_l(k) = 2k + 2l$. Later, Petingi and Talafha [159] improved "$2k + 2l$" to "$2k + l + 2$". Jain, Mahdian ans Salavatipour [108] proved $f_l(k) = \frac{l}{4} + o(l)$.

Frank, Király and Kriesell [83] showed that if $V(G) - S$ is an independent set, then the $3k$-edge-connectivity of S guarantees the existence of k edge-disjoint S-Steiner trees. Kriesell [124] improved this result and obtained that the $(m + 2)k$-edge-connectivity guarantees the existence of k edge-disjoint S-Steiner trees, where m is the order of the largest component of $G - S$. Lau [128] showed that Conjecture 46 holds if S is $26k$-edge-connected. However, Conjecture 46 is still open.

In the rest of this section, we explain some applications of edge-disjoint spanning trees. In particular, the existence of two edge-disjoint spanning trees implies some good properties. For a graph G, a set of cycles is called *cycle double cover of G* if every edge of G is contained in exactly two cycles of them. Seymour and Szekeres, independently, posed the following famous conjecture.

Conjecture 47 (Seymour [169], Szekeres [173]) *Let G be a 2-edge-connected graph. Then G has a cycle double cover.*

Jaeger [107] showed that this conjecture is true for 4-edge-connected graphs. We can show that every graph having two edge-disjoint spanning trees also has a cycle double cover, and hence Jaeger's result is implied by Corollary 44. (See also [122].)

On the other hand, the existence of two edge-disjoint spanning trees implies the existence of a spanning connected Eulerian subgraph. This fact concerns the following conjecture.

Conjecture 48 (Matthews and Sumner [141]) *Every 4-connected claw-free graph is hamiltonian.*

Zhan [190], and independently Jackson [105] proved that Conjecture 48 is true for 7-connected claw-free graphs. In fact, they proved it by showing the existence of two edge-disjoint spanning trees.

7.3 Independent spanning trees

Let G be a connected graph and let $T_1, T_2, \ldots, T_k$ be spanning trees with root r of G. We say that T_1 and T_2 are *independent* if two paths vT_1r and vT_2r are internally vertex-disjoint for every vertex v. When T_i and T_j are independent for all $1 \leq i < j \leq k$, $T_1, T_2, \ldots, T_k$ are called *independent*.

If G has k independent spanning trees with root r for all $r \in V(G)$, then G is k-connected because there exist k internally disjoint paths between every two vertices.

Itai and Rodeh conjectured that the converse also holds.

Conjecture 49 (Itai and Rodeh [104]) *Let G be a k-connected graph and let $r \in V(G)$. Then G has k independent spanning trees with root r.*

Conjecture 49 is still open, but we have some partial solutions. Itai and Rodeh [104] showed that it is true for the case $k = 2$ by themselves. The case $k = 3$ of Conjecture 49 was also shown by Zehavi and Itai [189], and by Cheriyan and Maheshwari [40], independently. Huck proved that Conjecture 49 is true for 4-connected planar graphs [100] and for 5-connected planar graphs [103]. For 4-connected planar graphs, Miura, Takahashi, Nakano and Nishizeki [143] gave a linear-time algorithm for finding four independent spanning trees. Recently, Curran, Lee and Yu [49] proved the case $k = 4$ for general graphs using Miura et al.'s algorithm together with an extension of ear decomposition, called *chain decomposition*. In [48, 50], they obtained a chain decomposition of a 4-connected graph. Conjecture 49 still remains open for $k \geq 5$.

On the other hand, there exist some variations of Conjecture 49. The digraph version of Conjecture 49 is mentioned as follows: For every k-connected digraph D and every vertex r in D, there exist k independent spanning directed trees with root r. This conjecture was verified by Whitty [179] for the case $k = 2$. However, this conjecture was, unfortunately, disproved by Huck [101] for the general digraph when $k \geq 3$. Therefore it has been considered for some restricted classes. For example, it is true for acyclic digraphs by Huck [102], for deBruijn and Kautz digraphs by Bermond and Fraigniaud [15], and for iterated line graphs by Hasunuma and Nagamochi [98].

When we define independent spanning trees, we consider internally "vertex"-disjoint paths. Similarly, we consider internally "edge"-disjoint paths. For two spanning trees T_1 and T_2 with root r, we call them *edge-independent* if two paths rT_1v and rT_2v are edge-disjoint for every vertex v. The following edge version of Conjecture 49 was also proposed by Itai and Rodeh [104]: For every k-edge-connected graph G and for every vertex r in G, there exist k edge-independent spanning trees with root r. Khuller and Schieber [119] proved that if Conjecture 49 (the vertex version) is true for some k, then the edge version conjecture is also true for k. Therefore, we obtain the edge version conjecture is true for the case $k \leq 4$ by the vertex version results as mentioned above. For $k \geq 5$, the edge version conjecture is also still open.

7.4 Graphs which are determined from spanning trees

Let G be a connected graph and let $\mathcal{T}(G)$ be the set of spanning trees of G. The *tree graph of* G, denoted by $T(G)$, is the graph such that the vertex set of $T(G)$ is $\mathcal{T}(G)$ and two vertices $T_1, T_2 \in \mathcal{T}(G)$ are adjacent in $T(G)$ if and only if $|E(T_1) \backslash E(T_2)| = 1$.

Cummins [46] proved that $T(G)$ is hamiltonian, so 2-connected. Shank [170] gave a short proof of it, and Liu [135] obtained a sharp value of the connectivity of the tree graph using the invariant concerning with the cycle space.

Sometimes we consider a restricted tree graphs. Let G be a connected graph and let $T_1, T_2 \in \mathcal{T}(G)$ with $E(T_1) \setminus E(T_2) = \{e_1\}$ and $E(T_2) \setminus E(T_1) = \{e_2\}$. Note that T_1 and T_2 are adjacent in the tree graph of G. The *adjacency tree graph* of the graph G is a graph on $\mathcal{T}(G)$ such that $T_1, T_2 \in \mathcal{T}(G)$ are adjacent if and only if e_1 and e_2 are adjacent in G. The *leaf-exchange tree graph* is the graph on $\mathcal{T}(G)$ such that $T_1, T_2 \in \mathcal{T}(G)$ are adjacent if and only if e_1 and e_2 are incident with a leaf of T_1 and T_2, respectively.

For a connected multigraph G, let ρ be the dimension of the cycle space, that is, $\rho := |E(G)| - |V(G)| + 1$. Zhang and Chen [191] proved that the adjacency tree graph of G is ρ-connected, and Heinrich and Liu [99] improved this result to 2ρ-conencted, when G is a simple graph. Estivill-Castro, Noy and Urrutia [75] studied the chromatic number of the tree graph and the adjacency tree graph.

Harary, Mokken and Plantholt [97] proved that the leaf-exchange tree graph of a 2-connected graph G is connected. Broersma and Li [25] characterized the graphs whose leaf-exchange tree graph is connected, and gave a lower bound on the connectivity of the leaf-exchange tree graph of a 3-connected graph. Kaneko and Yoshimoto [113] proved that the leaf-exchange tree graph of a 2-connected graph G is $(2\delta(G) - 2)$-connected.

Recently, Li, Neumann-Lara and Rivera-Campo [132] defined a new graph on $\mathcal{T}(G)$. For a set $\mathcal{C}$ of cycles of a connected graph G, $T(G, \mathcal{C})$ is defined as a spanning subgraph of the tree graph of G such that $T_1, T_2 \in \mathcal{T}(G)$ are adjacent if and only if the unique cycle of $T_1 \cup T_2$ is contained in $\mathcal{C}$. They also gave some necessary conditions and sufficient conditions for $T(G, \mathcal{C})$ to be connected.

Yoshimoto [187] proposed other graph on $\mathcal{T}(G)$. The *trunk graph* of G is the graph on $\mathcal{T}(G)$ such that $T_1, T_2 \in \mathcal{T}(G)$ are adjacent if and only if e_1 and e_2 are not adjacent to a leaf of T_1 and a leaf of T_2, respectively. He showed that the trunk graph of a 2-connected k-edge-connected graph of order at least five is $(k-1)$-connected.

Acknowledgements

The authors would like to thank the referees for their helpful comments and corrections.

References

[1] N. Alon, Transversal numbers of uniform hypergraphs, Graphs Combin. **6** (1990) 1–4.

[2] N. Alon, F.V. Fomin, G. Gutin, M. Krivelevich and S. Saurabh, Parameterized algorithms for directed maximum leaf problems, Lecture Notes in Computer Science **4596** (2007) 352–362.

[3] N. Alon, F.V. Fomin, G. Gutin, M. Krivelevich and S. Saurabh, Better algorithms and bounds for directed maximum leaf problems, Lecture Notes in Computer Science **4855** (2007) 316–327.

[4] N. Alon and N. Wormald, High degree graphs contain large-star factors, arXiv:math.CO.0810.2053v1.

[5] D. Archedeacon, N. Hartsfield and C.H.C. Little, Nonhamiltonian triangulations with large connectivity and representativity, J. Combin Theory Ser. B **68** (1996) 45–55.

[6] T. Atajan, X. Yong and H. Inaba, Further analysis of the number of spanning trees in circulant graphs, Discrete Math. **306** (2006) 2817–2827.

[7] M. Aung and A. Kyaw, Maximal trees with bounded maximum degree in a graph, Graphs and Combin. **14** (1998) 209–221.

[8] T.I. Austin, The enumeration of point labelled chromatic graphs and trees, Canad. J. Math. **12** (1960) 535–545.

[9] C. S. Babu and A.A. Diwan, Degree conditions for forests in graphs, Discrete Math. **301** (2005) 228–231.

[10] D.W. Barnette, Trees in polyhedral graphs, Canad. J. Math **18** (1966) 731–736.

[11] D.W. Barnette, 3-trees in polyhedral maps, Israel. J. Math **79** (1992) 251–256.

[12] D. Bauer, H. Broersma and E. Schmeichel, Toughness in graphs - a survey, Graphs and Combin. **22** (2006) 1–35.

[13] D. Bauer, H. Broersma and H.J. Veldman, Not every 2-tough graph is hamiltonian, Discrete Appl. Math. **99** (2000) 317–321.

[14] D. Bauer, G. Fan and H.J. Veldman, Hamiltonian properties of graphs with large neighborhood unions, Discrete Math. **96** (1991) 33–49.

[15] J.-C. Bermond and P. Fraigniaud, Broadcasting and gossiping in deBruijn networks, SIAM J. Comput. **23** (1994) 212–225.

[16] F. Boesch, X. Li and C. SuLel, On the existence of uniformly most reliable networks, Networks **21** (1991) 181–194.

[17] T. Böhme, H.J. Broersma, F. Göbel, A.V. Kostochka and M. Stiebitz, Spanning trees with pairwise nonadjacent endvertices, Discrete Math. **170** (1997) 219–222.

[18] A. Bondy and V. Chvátal, A method in graph theory, Discrete Math. **15** (1976) 111–135.

[19] P. Bonsma, Spanning trees with many leaves in graphs with minimum degree three, SIAM Journal on Discrete Mathematics, **22** (3) (2008) 920–937.

[20] P. Bonsma and F. Zickfeld, Spanning Trees with Many Leaves in Graphs without Diamonds and Blossoms, Lecture Notes in Computer Science **4957** (2008) 531–543.

[21] P. Bonsma and F. Zickfeld, A 3/2-approximation algorithm for finding spanning trees with many leaves in cubic graphs, Lecture Notes in Computer Science **5344** (2008) 66–77.

[22] S. Brandt, Subtrees and Subforests of Graphs, J. Combin. Theory Ser. B **61** (1994) 63–70.

[23] M.F. Bridgland, R.E. Jamison and J.S. Zito, The spanning trees forced by the path and the star, J. Graph Theory **23** (1996) 421–441.

[24] H. Broersma, O. Koppius, H. Tuinstra, A. Huck, T. Kloks, D. Kratsch and H. Müller, Degree-preserving trees, Networks **35** (2000) 26–39.

[25] H.J. Broersma and X. Li, The connectivity of the leaf-exchange spanning tree graph of a graph, Ars Comnin. **43** (1996) 225–231.

[26] H. Broersma and H. Tuinstra, Independence trees and Hamilton cycles, J. Graph Theory **29** (1998) 227–237.

[27] L. Cai, On spanning 2-trees in a graph, Discret Appl. Math. **74** (1997) 203–216.

[28] L. Cai, The complexity of the locally connected spanning tree problem, Discret Appl. Math. **131** (2003) 63–75.

[29] Y. Caro, I. Krasikov and Y. Roditty, On the largest tree of given maximum degree in a connected graph, J. Graph Theory **15** (1991) 7–13.

[30] Y. Caro, D.B. West, and R. Yuster, Connected domination and spanning trees with many leaves, SIAM J. Discrete Math. **13** (2000) 202-211.

[31] P.A. Catlin, Edge-connectivity and edge-disjoint spanning trees, See: `http://www.math.wvu.edu/~hjlai/Pdf/Catlin_Pdf/Catlin49a.pdf` preprint.

[32] P.A. Catlin, J.W. Grossman, A.M. Hobbs and H.J. Lai, Fractional arboricity, strength, and principal partitions in graphs and matroids, Discrete Appl. Math. **40** (1992) 285–302.

[33] P.A. Catlin, H.J. Lai and Y. Shao, Edge-connectivity and edge-disjoint spanning trees, Discrete Math. **309** (2009) 1033–1040.

[34] A. Cayley, A theorem on trees, Quart. J. Math. **23** (1889) 376–378.

[35] G. Chen, Y. Egawa, K. Kawarabayashi, B. Mohar and K. Ota, Toughness of $K_{a,t}$-minor-free graphs, submitted.

[36] C.C. Chen, K.M. Koh and Y.H. Peng, On the higher-order edge toughness of a graph, Discrete Math. **111** (1993) 113–123.

[37] Z.H. Chen and H.J. Lai, The higher-order edge-toughness of a graph and truncated uniformly dense matroids, J. Combin. Math. Combin. Comput. **22** (1996) 157–160.

[38] X. Chen, Q. Lin and F. Zhang, The number of spanning trees in odd valent circulant graphs, Discrete Math. **282** (2004) 69–79.

[39] C. Cheng, Maximizing the total number of spanning trees in a graph: two related problems in graph theory and optimization design theory, J. Combin. Theory Ser. B **31** (1981) 240–248.

[40] J. Cheriyan and S. Maheshwari, Finding nonseparating induced cycles and independent spanning trees in 3-connected graphs, J. Algorithms **9** (1988) 507–537.

[41] S. Choi and P. Guan, A spanning tree of the 2^m-dimensional hypercube with maximum number of degree-preserving vertices, Discrete Math. **117** (1993) 275–277.

[42] V. Chvàtal, Tough graphs and hamiltonian circuits, Disceret Math. **5** (1973) 215–228.

[43] V. Chvàtal, Tree–complete graph Ramsey numbers, J. Graph Theory **1** (1977) 93.

[44] V. Chvátal and P. Erdős, A note on hamiltonian circuits, Discrete Math. **2** (1972) 111–113.

[45] J.R. Correa, C.G. Fernandes, M. Matamala and Y. Wakabayashi, A 5/3-approximation for finding spanning trees with many leaves in cubic graphs, Lecture Notes in Computer Science **4927** (2008) 184–192.

[46] R.L. Cummins, Hamilton circuits in tree graphs, IEEE Trans. Circuit Theory **13** (1966) 82–90.

[47] W.H. Cunningham, Optimal attack and reinforcement of a network, J. Assoc. Comput. Mach. **32** (1985) 549–561.

[48] S. Curran, O. Lee and X. Yu, Chain decompositions of 4-connected graphs, SIAM J. Discrete Math. **19** (2005) 848–880.

[49] S. Curran, O. Lee and X. Yu, Finding four independent trees, SIAM J. Comput. **35** (2006) 1023–1058.

[50] S. Curran, O. Lee and X. Yu, Nonseparating planar chains in 4-connected graphs, SIAM J. Discrete Math. **19** (2006) 399–419.

[51] D. Cvetkovič, M. Doob and H. Sachs, Spectra of Graphs, Mathematics 87, Academic press, New York 1980.

[52] A. Czygrinow, G. Fan, G. Hurlbert, H.A. Kierstead and W.T. Trotter, Spanning trees of bounded degree, Electronic Journal of Combinatorics **8** (2001) R33.

[53] E. Dahlhaus, P. Dankelmann, W. Goddard and H.C. Swart, MAD trees and distance-hereditary graphs, Dicrete Appl. Math. **131** (2003) 151–167.

[54] E. Dahlhaus, P. Dankelmann and R. Ravi, A linear-time algorithm to compute a MAD tree of an interval graph, Inform. Process. Lett. **89** (2004) 255–259.

[55] P. Dankelmann and R. Entringer, Average distance, minimum degree, and spanning trees, J. Graph Theory **33** (2000) 1–13.

[56] P. Damaschke, Degree-preserving spanning trees in small-degree graphs, Discrete Math. **222** (2000) 51–60.

[57] K.C. Das, A sharp upper bounds for the number of spanning trees of a graph, Graphs and Combin. **23** (2007) 625–632.

[58] G. Ding, T. Johnson and P. Seymour, Spanning trees with many leaves, J. Graph Theory **37** (2001) 189–197.

[59] Y. Egawa, H. Matsuda, T. Yamashita and K. Yoshimoto, On a spanning tree with specified leaves, Graphs Combin. **24** (2008) 13–18.

[60] Y. Egawa and K. Ozeki, A necessary and sufficient condition for the existence of a spanning tree with specified vertices having large degrees, submitted.

[61] O. Eğecioğlu and J.B. Remmel, Bijections for Cayley trees, spanning trees and their q-analogues, J. Combin. Theory Ser. A **42** (1986) 15–30.

[62] O. Eğecioğlu and J.B. Remmel, A bijection for spanning trees of complete multipartite graphs, Congr. Numer. **100** (1994) 225–243.

[63] M.N. Ellingham, Spanning paths, cycles and walks for graphs on surfaces, Congressus Numerantium **115** (1996) 55–90.

[64] M.N. Ellingham and Z. Gao, Spanning trees in locally planar triangulations, J. Combin Theory Ser. B **61** (1994) 178–198.

[65] M.N. Ellingham, Y. Nam and H.-J. Voss, Connected (g, f)-factor, J. Graph Theory **39** (2002) 62–75.

[66] M.N. Ellingham and X. Zha, Toughness, trees, and walks, J. Graph Theory **33** (2000) 125–137.

[67] H. Enomoto, private communication.

[68] H. Enomoto, Y. Ohnishi and K. Ota, Spanning trees with bounded total excess, to appear in Ars Combin.

[69] H. Enomoto and K. Ozeki, The independence number condition for the existence of a spanning f-tree, to appear in J. Graph Thoery.

[70] R.C. Entringer, Distance in graphs: trees, J. Combin. Math. Combin. Comput. **24** (1997) 65–84.

[71] R.C. Entringer, D.J. Kleitman, and L.A. Székely, A note on spanning trees with minimum average distance, Bull. Inst. Combim. Appl. **17** (1996) 71–78.

[72] P. Erdős, Extremal problems in graph theory, in "Theory of Graphs and its Applications", Academic Press, New York, (1964) 29–36.

[73] P. Erdős, R.J. Faudree, C.C. Rousseau and R.H. Schelp, Graphs with certain families of spanning trees, J. Combin. Theory Ser. B **32** (1982) 162–170.

[74] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar **10** (1959) 337–356.

[75] V. Estivill-Castro, M. Noy and J. Urrutia, On the chromatic number of tree graphs, Discrete Math. **223** (2000) 363–366.

[76] G. Fan and L. Sun, The Erdős-Sós conjecture for spiders, Discrete Math. **307** (2007) 3055–3062.

[77] R.J. Faudree, C.C. Rousseau, R.H. Schelp and S. Schuster, Panarboreal graphs, Israel J. Math. **35** (1980) 177–185.

[78] L. Feng, G. Yu, Z. Jiang and L. Ren, Sharp upper bounds for the number of spanning trees of a graph, Appl. Anal. Discrete Math. **2** (2008) 255–259.

[79] M. Fischetti, G. Lancia and P. Serafini, Exact algorithms for minimum routing cost trees, Networks **39** (2002) 161–173.

[80] E. Flandrin, H.A. Jung and H. Li, Hamiltonism, degree sum and neighborhood intersections, Discrete Math. **90** (1991) 41–52

[81] E. Flandrin, T. Kaiser, R. Kužel, H. Li and Z. Ryjáček, Neighborhood unions and extremal spanning trees, Discrete Math. **308** (2008) 2343–2350.

[82] A. Frank and A. Gyàrfàs, How to orient the edges of a graph? Colloq. Math. Soc. Jànos Bolyai **18** (1976) 353–364.

[83] A. Frank, T. Király and M. Kriesell, On decomposing a hypergraph into k connected sub-hypergraphs, Discrete Appl. Math. **131** (2003) 373–383.

[84] J. Fujisawa, H. Matsumura and T. Yamashita, Degree bounded spanning trees, to appear in Graphs and Combin.

[85] J. Fujisawa, A. Saito and I. Schiermeyer, Closure for spanning trees and distant area, submitted.

[86] E.G. Fusco and A. Monti, Spanning trees with many leaves in regular bipartite graphs, Lecture Notes in Computer Science **4835** (2007) 904–914.

[87] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, Freeman, New York, 1979.

[88] L. Gargano and M. Hammar, There are spanning spiders in dense graphs (and we know how to find them), Lecture Notes in Comput. Sci. **2719** (2003) 802–816.

[89] L. Gargano, M. Hammar, P. Hell, L. Stacho and U. Vaccaro, Spanning spiders and light-splitting switches, Discrete Math. **285** (2004) 83–95.

[90] L. Gargano, P. Hell, L. Stacho and U. Vaccaro, Spanning trees with bounded number of branch vertices, Lecture Notes in Comput. Sci. **2380** (2002) 355–365.

[91] B. Gilbert and W. Myrvold, Maximizing spanning trees in almost complete graphs, Networks **30** (1997) 23–30.

[92] J.R. Griggs, D.J. Kleitman and A. Shastri, Spanning trees with many leaves in cubic graphs, J. Graph Theory **13** (1989) 669–695.

[93] J.R. Griggs and M. Wu, Spanning trees in graphs of minimum degree 4 or 5, Discrete Math. **104** (1992) 167–183.

[94] G.R. Grimmett, An upper bound for the number of spanning trees of a graph, Discrete Math. **16** (1976) 323–324.

[95] R. Grone and R. Merris, A bound for the complexity of a simple graph, Discrete Math. **69** (1988) 97–99.

[96] M.A. Gurgel and Y. Wakabayashi, On k-leaf-connected graphs, J. Combin. Theory Ser. B **41** (1986) 1–16.

[97] F. Harary, R.J. Mokken abd M.J. Plantholt, Interpolation theorem for diameters of spanning trees, IEEE Trans. Circuits Systems **30** (1983) 429–432.

[98] T. Hasunuma and H. Nagamochi, Independent spanning trees with small depths in iterated line graphs, Discrete Appl. Math. **110** (2001) 189–211.

[99] K. Heinrich and G. Liu, A lower bound on the number of spanning trees with k end-vertices, J. Graph Theory **12** (1988) 95–100.

[100] A. Huck, Independent trees in graphs, Graphs and Combim. **10** (1994) 29–45.

[101] A. Huck, Disproof of a conjecture about independent spanning trees in k-connected directed graphs, J. Graph Theory **20** (1995) 235–239.

[102] A. Huck, Independent branching in acyclic digraphs, Discrete Math. **199** (1999) 245–249.

[103] A. Huck, Independent trees in planar graphs independent trees, Graphs and Combim. **15** (1999) 29–77.

[104] A. Itai and M. Rodeh, The multi-tree approach to reliability in distributed networks, Inform. and Comput. **79** (1988) 43–59.

[105] B. Jackson, Hamilton cycles in 7-connected line graphs, preprint.

[106] B. Jackson and N.C. Wormald, k-walks of graphs, Australasin J. Combin. **2** (1990) 135–146.

[107] F. Jaeger, Flows and generalized coloring theorems in graphs, J. Combin. Theory Ser. B **26** (1979) 205–216.

[108] K. Jain, M. Mahdian and M.R. Salavatipour, Packing Steiner trees, Proceedings of the fourteenth annual ACM-SIAM symposium on Discrete algorithms (SODA), (2003) 266–274.

[109] D.S. Jhonson, J.K. Lenstra and A.H. Rinnooy-Kan, The complexity of the network design problem, Networks **8** (1978) 279–285.

[110] T. Jordán, On the existence of k edge-disjoint 2-connected spanning subgraphs, J. Combin. Theory Ser. B **95** (2005) 257–262.

[111] A. Kaneko, Spanning trees with constraints on the leaf degree, Discrete Appl. Math. **115** (2001) 73–76.

[112] A. Kaneko, M. Kano and K. Suzuki, Spanning trees with leaf distance at least four, J. Graph Theory **55** (2007) 83–90.

[113] A. Kaneko and K. Yoshimoto, The connectivities of leaf graphs of 2-connected graphs, J. Combin. Theory Ser. B **76** (1999) 155–169.

[114] A. Kaneko and K. Yoshimoto, On spanning trees with restricted degrees, Inform. Process. Lett. **73** (2000) 163–165.

[115] M. Kano and H. Kishimoto, Spanning k-tree of n-connected graphs, to appear in Graphs and Combin.

[116] M. Kano, Aung Kyaw, H. Matsuda, K. Ozeki, A. Saito and T. Yamashita, Spanning trees with a bounded number of leaves in a claw-free graph, submitted.

[117] K. Kawarabayashi, A. Nakamoto and K. Ota, Subgraphs of graphs on surfaces with high representativity, J. Combin. Theory Ser. B **89** (2003) 207–229.

[118] A. Kelmans, On graphs with the maximum number of spanning trees, Random Struct. Algorithms **9** (1996) 177–192.

[119] B. Khuller and B. Schieber, On independent spanning trees, Inform. Process. Lett. **42** (1992) 321–323.

[120] G. Kirchhoff, Über die Auflösung der Gleichungen, aufwelche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, Ann Phys. Chem. **72** (1847) 497–508.

[121] D.J. Kleitman and D.B. West, Spanning trees with many leaves, SIAM J. Discrete Math. **4** (1991) 99–106.

[122] M. Kouider and P.D. Vestergaard, Connected factors in graphs – a survey, Graphs and Combin. **21** (2005) 1–26.

[123] M. Kriesell, Edge-disjoint trees containing some given vertices in a graph, J. Combin. Theory Ser. B **88** (2003) 53–65.

[124] M. Kriesell, Edge disjoint Steiner trees in graphs without large bridges, J. Graph Theory **62** (2009) 188–198.

[125] A. Kyaw, A sufficient condition for a graph to have a k-tree, Graphs Combin. **17** (2001) 113–121.

[126] A. Kyaw, Spanning trees with at most 3 leaves in $K_{1,4}$-free graphs, Discrete Math. **309** (2009) 6146–6148.

[127] A. Kyaw, private communication.

[128] L.C. Lau, An approximate max-Steiner-tree-packing min-Steiner-cut theorem, Combinatorica **27** (2007) 71–90.

[129] P. Lemke, The maximum leaf spanning tree problem for cubic graphs is NP-complete, IMA Preprint Series, # 428, 1988.

[130] M. Lewinter, Interpolation theorem for the number of degree-preserving vertices of spanning trees, IEEE Trans. Circuits and Systems **34** (1987) 205.

[131] R.P. Lewis, The number of spanning trees of a complete multipartite graph, Discrete Math. **197/198** (1999) 537–541.

[132] X. Li, V. Neumann-Lara and E. Rivera-Campo, On a tree graph defined by a set of cycles, Discrete Math. **271** (2003) 303–310.

[133] P.C. Li and M. Toulouse, Variations of the maximum leaf spanning tree problem for bipartite graphs, Inform. Process. Lett. **97** (2006) 129–132

[134] C.C. Lin, G.J. Chang and G.H. Chen, Locally connected spanning trees in strongly chordal graphs and proper circular graphs, Discrete Math. **307** (2007) 208–215.

[135] G.Z. Liu, On connectivities of tree graphs, J. Graph Theory **12** (1988) 453–459.

[136] K. Loryś and G. Zwoźniak, Approximation algorithm for the maximum leaf spanning tree problem for cubic graphs, Lecture Notes in Computer Science **2461** (2002) 686–697.

[137] H.I. Lu and R. Ravi, The power of local optimization: approximation algorithms for maximum-leaf spanning tree, in "Proceedings of the Thirtieth Annual Allerton Conference on Communication, Control and Computing," (1992) 533–542.

[138] H.I. Lu and R. Ravi, Approximating maximum leaf spanning trees in almost linear time, J. Algorithms **29** (1998) 132–141.

[139] H. Matsuda and H. Matsumura, On a k-tree containing specified leaves in a graph, Graphs and Combin. (2006) **22** 371–381

[140] H. Matsuda, K. Ozeki and T. Yamashita, Spanning trees with a bounded number of branch vertices in a claw-free graph, submitted.

[141] M.M. Matthews and D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory **8** (1984) 139–146.

[142] R. Merris, Laplacian Matrices of Graphs: A survey, Linear Algebra Appl. **197/198** (1994) 143–176.

[143] K. Miura, D. Takahashi, S. Nakano and T. Nishizeki, A linear-time algorithm to find four independent spanning trees in four-connected planar graphs, Lecture Notes in Computer Science, Graph-Theoretic Concepts in Computer Science **1517** (1998) 310–323.

[144] J.W. Moon, Counting labelled trees, Canad. Math. Congress, 1970.

[145] A. Nakamoto, Y. Oda and K. Ota, 3-trees with few vertices of degree 3 in circuit graphs, Discrete Math. **309** (2009) 666–672.

[146] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. **36** (1961) 445–450.

[147] V. Neumann-Lara and E. Rivera-Campo, Spanning trees with bounded degrees, Combinatorica **11** (1991) 55–61.

[148] Y. Ohnishi and K. Ota, Connected factors with bounded total excess, preprint.

[149] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly **67** (1960) 55.

[150] O. Ore, Hamilton connected graphs, J. Math. Pures Appl. **42** (1963) 21–27.

[151] K. Ota and K. Ozeki, Spanning trees in 3-connected $K_{3,t}$-minor-free graphs, submitted.

[152] K. Ozeki, Toughness condition for a spanning k-tree with bounded total excess, submitted.

[153] K. Ozeki, Spanning trees in 3-connected graphs on surfaces, preprint.

[154] K. Ozeki and T. Yamashita, A spanning tree with high degree vertices, Graphs and Combin. **26** (2010) 591–596.

[155] E.M. Palmer, On the spanning tree packing number of a graph: a survey, Discrete Math. **230** (2001) 13–21.

[156] L. Petingi, F. Boesch and C. SuLel, On the characterization of graphs with maximum number of spanning trees, Discrete Math. **179** (1998) 155–166.

[157] L. Petingi and J. Rodriguez, Bounds on the maximum number of edge-disjoint Steiner trees of a graph, Congressus Numerantium **145** (2000) 43–52.

[158] L. Petingi and J. Rodriguez, A new technique for the characterization of graphs with a maximum number of spanning trees, Discrete Math. **244** (2002) 351–373.

[159] L. Petingi and M. Talanfha, Packing the Steiner trees of a graph, Networks **54** (2009) 90–94.

[160] J. Plesnik, On the sum of all distances in a graph or digraph, J. Graph Theory **8** (1984) 1–21.

[161] M.D. Plummer, Graph factors and factorization: 1985–2003: a survey, Discrete Math. **307** (2007) 791–821.

[162] H. Prüfer, Neuer Beweis eines Satzes über Permutationen, Arch. Math. Phys. **27** (1918) 142–144.

[163] M.S. Rahman and M. Kaykobad, Complexities of some interesting problems on spanning trees, Inform. Process. Lett. **94** (2005) 93–97.

[164] E. Rivera-Campo, An Ore-type condition for the existence of spanning trees with bounded degrees, Congr. Numer. **90** (1992) 19–32.

[165] E. Rivera-Campo, A note on matchings and spanning trees with bounded degrees, Graphs Combin. **13** (1997) 159–165.

[166] G. Salamon, Approximation algorithms for the maximum internal spanning tree problem, Lecture Notes in Computer Science **4708** (2007) 90–102.

[167] G. Salamon and G. Wiener, On finding spanning trees with few leaves, Inform. Process. Lett. **105** (2008) 164–169.

[168] D.P. Sanders and Y. Zhao, On spanning trees and walks of low maximum degree, J. Graph Theory **36** (2001) 67–74.

[169] P.D. Seymour, Sums of circuits, Graph Theory and Relation Topics, Academic Press, New York, (1979) 341–355.

[170] H. Shank, A note on hamilton circuits in tree graphs, IEEE Trans. Circuit Theory **15** (1968) 86.

[171] R. Solis-Oba, 2-approximation algorithm for finding a spanning tree with maximum number of leaves, Lecture Notes in Computer Science **1461** (1998) 441–452.

[172] J. Szabó, Packing trees with constraints on the leaf degree, Graphs Combin. **24** (2008) 485–494.

[173] G. Szekeres, Polyhedral decomposition of cubic graphs, Bull. Austral. Math. Soc. **8** (1973) 367–387.

[174] C. Thomassen, Trees in triangulations, J. Combin. Theory Ser. B **60** (1994) 56–62.

[175] M. Tsugaki, A note on a spanning 3-tree, Combinatorica **29** (2009) 127–129.

[176] M. Tsugaki and T. Yamashita, Spanning trees with few leaves, Graphs Combin. **23** (2007) 585–598.

[177] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math.Soc. **36** (1961) 221–230.

[178] D.B. West, Introduction to Graph Theory, Second Edition, Prentice-Hall, Inc. 1996.

[179] R.W. Whitty, Vertex-disjoint paths and edge-disjoint branchings in directed graphs, J. Graph Theory **11** (1987) 349–358.

[180] S. Win Existenz von Gerüsten mit vorgeschriebenem Maximalgrad in Graphen (German), Abh. Math. Sem. Univ. Hamburg **43** (1975) 263–267.

[181] S. Win, On a conjecture of Las Vergnas concerning certain spanning trees in graphs, Resultate Math. **2** (1979) 215–224.

[182] S. Win, On a connection between the existence of k-trees and the toughness of a graph, Graphs and Combin. **5** (1989) 201–205.

[183] R. Wong, Worst-case analysis of network design problem heuristics, SIAM J. Algebraic Discrete Math. **1** (1980) 51–63.

[184] B.Y. Wu, K.M. Chao and C.Y. Tang, Approximation algorithms for the shortest total path length spanning tree problem, Discrete Appl. Math. **105** (2000) 273–289.

[185] B.Y. Wu, G. Lancia, V. Bafna, K.M. Chao, R. Ravi and C.Y. Tang, A polynomial-time approximation scheme for minimum routing cost spanning trees, SIAM J. Comput. **29** (2000) 761–778.

[186] X.R. Yong and T. Acenjian, The numbers of spanning trees of the cycle C_N^3 and the quadruple cycle C_N^4, Discrete Math. **169** (1997) 293–298.

[187] K. Yoshimoto, The connectivities of trunk graphs of 2-connected graphs, Ars Combin. **60** (2001) 225–237.

[188] X. Yu, Disjoint paths, planarizing cycles, and spanning walks, Trans. Amer. Math. Soc. **349** (1997) 1333–1358.

[189] A. Zehavi and A. Itai, Three tree-paths, J. Graph Theory **13** (1989) 175–188.

[190] S.M. Zhan, On Hamiltonian line graphs and connectivity, Discrete Math. **89** (1991) 89–95.

[191] F.J. Zhang and Z. Chen, Connectivity of (adjacency) tree graphs, J. Xinjiang Univ. Nat. Sci. **3** (1986) 1–5.

[192] Y. Zhang, X. Yong and M.J. Golin, The number of spanning trees in circulant graphs, Discrete Math. **223** (2000) 337–350.

[193] L. Zhenhong and X. Baoguang, On low bound of degree sequences of spanning trees in k-edge-connected graphs, J. Graph Theory **28** (1998) 87–95.

[194] I.A. Zioło, Subforests of bipartite figraphs – the minimum degree condition, Discrete Math. **236** (2001) 351–365.

[195] I.A. Zioło, Subtrees of bipartite figraphs – the minimum degree condition, Discrete Appl. Math. **99** (2000) 251–259.