Length of longest cycles in a graph whose relative length is at least two

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Abstract

Let G be a graph. We denote $p(G)$ and $c(G)$ the order of a longest path and the order of a longest cycle of G, respectively. Let $\kappa(G)$ be the connectivity of G, and let $\sigma_3(G)$ be the minimum degree sum of an independent set of three vertices in G. In this paper, we prove that if G is a 2-connected graph with $p(G) - c(G) \geq 2$, then either (i) $c(G) \geq \sigma_3(G) - 3$ or (ii) $\kappa(G) = 2$ and $p(G) \geq \sigma_3(G) - 1$. This result implies several known results as corollaries and gives a new lower bound of the circumference.

Keywords: circumference, relative length, degree sum

1 Introduction

Cycle-related properties of graphs have been studied for a long time. As one of them, we consider the length of a longest cycle, called *circumference*. Many researchers have estimated the lower bound of the circumference of graphs by various invariants; minimum degree ([6, 7]), minimum degree and girth ([8, 17, 21]), minimum degree and toughness ([1, 13]), neighborhood union ([10, 16]) and so on. Good references for circumference are [4, 20].

The purpose of this paper is to estimate the circumference of a graph G by two invariants $\sigma_3(G)$ and $p(G)$. For a graph G, let $\sigma_k(G)$ be the minimum degree sum of k independent vertices of G if $\alpha(G) \geq k$; otherwise we let $\sigma_k(G) = +\infty$, where $\alpha(G)$ is the independence number of G. Let $p(G)$ be the order of a longest path of a graph G.

Let $c(G)$ be the order of a longest cycle of a graph G. Note that $c(G)$ is equal to the circumference of a graph G. Bermond [2] and Linial [15], independently, proved that $c(G) \geq \min\{\sigma_2(G), n\}$ for every 2-connected graph G on n vertices. Fournier and Fraisse showed a generalization of this result, conjectured by Bondy [3]; if G is a k-connected graph ($k \geq 2$) on n vertices, then $c(G) \geq \min\{2\sigma_{k+1}(G)/(k+1), n\}$. As mentioned above, in this paper, we are interested in $\sigma_3(G)$, that is, the case $k = 2$ of this result.

Theorem 1 (Fournier and Fraisse [11]) *Let G be a 2-connected graph on n vertices. Then $c(G) \geq \min\{2\sigma_3(G)/3, n\}$.*

On the other hand, Dirac [6] showed that $c(G) \geq 2(p(G) - 1)^{1/2}$ for a 2-connected graph G. Bondy and Locke improved this result for 3-connected graphs. (We do not know whether the coefficient $2/5$ is sharp, but it is well-known that it is not greater than $3/4$. For example, consider the graph $K_3 + 4K_m$ for some integer m, where "+" means the join of two graphs.)

Theorem 2 (Bondy and Locke [5]) *Let G be a 3-connected graph. Then $c(G) \geq 2(p(G) - 1)/5$.*

Motivated by Theorems 1 and 2, we estimate $c(G)$ by $\sigma_3(G)$ and $p(G)$ in 3-connected graphs, and prove the following result.

Theorem 3 *Let G be a 3-connected graph. Then $c(G) \geq \min\{\sigma_3(G) - 3, p(G) - 1\}$.*

This result is linked to the research on the invariant $\text{diff}(G) := p(G) - c(G)$, and can be rewritten in terms of $\text{diff}(G)$; $c(G) \geq \sigma_3(G) - 3$ or $\text{diff}(G) \leq 1$ for a 3-connected graph G. Many articles have been devoted to study the invariant $\text{diff}(G)$, because a graph G with small $\text{diff}(G)$ has a number of cycle-related properties. (See [9, 14, 18].) For example, a graph G is hamiltonian if and only if $\text{diff}(G) = 0$, and it is easy to see that every longest cycle of a graph G is dominating if $\text{diff}(G) \leq 1$. A cycle C of a graph G is *dominating* if $V(G \setminus C)$ is an independent set in G. Hence Theorem 3 implies Fraisse and Jung's theorem.

Corollary 4 (Fraisse and Jung [12]) *Let G be a 3-connected graph. Then $c(G) \geq \sigma_3(G) - 3$ or every longest cycle in G is dominating.*

Moreover, it is also known that a graph G with small $\mathrm{diff}(G)$ has a long cycle. Li et al. [14] showed that if $\mathrm{diff}(G) \leq 1$ for a 2-connected graph G on n vertices, then $c(G) \geq \min\{n - \alpha(G) + \delta(G), n\}$, where $\delta(G)$ is the minimum degree of a graph G. By this result, we obtain another corollary of Theorem 3. This gives a new lower bound of the circumference of 3-connected graphs.

Corollary 5 *Let G be a 3-connected graph on n vertices. Then $c(G) \geq \min\{\sigma_3(G) - 3, n - \alpha(G) + \delta(G), n\}$.*

In this paper, we prove a slightly stronger result than Theorem 3. We let $\kappa(G)$ denote the connectivity of a graph G.

Theorem 6 *Let G be a 2-connected graph. Then (i) $\mathrm{diff}(G) \leq 1$, or (ii) $c(G) \geq \sigma_3(G) - 3$, or (iii) $\kappa(G) = 2$ and $p(G) \geq \sigma_3(G) - 1$.*

Theorem 6 implies the following two corollaries.

Corollary 7 (Enomoto et al. [9]) *Let G be a 2-connected graph on n vertices. If $\sigma_3(G) \geq n + 2$, then $\mathrm{diff}(G) \leq 1$.*

Proof. Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n + 2$. We may assume that conclusion (ii) or (iii) in Theorem 6 holds. If (ii) holds, then $c(G) \geq \sigma_3(G) - 3 \geq n - 1$. Since $p(G) \leq n$, we obtain $\mathrm{diff}(G) = p(G) - c(G) \leq n - (n-1) = 1$. If (iii) holds, then $p(G) \geq \sigma_3(G) - 1 \geq n + 1$, a contradiction. □

Corollary 8 (Saito [19]) *Let G be a 2-connected graph with $\mathrm{diff}(G) \geq 2$. Then $p(G) \geq \sigma_3(G) - 1$.*

Proof. Let G be a 2-connected graph with $\mathrm{diff}(G) \geq 2$. By Theorem 6, we may assume that $c(G) \geq \sigma_3(G) - 3$. Then $p(G) = c(G) + \mathrm{diff}(G) \geq \sigma_3(G) - 1$. □

In the rest of this section, we will show that Theorem 6 is best possible in a sense. First, we let l, m be integers with $l \geq m + 1 \geq 5$, and let $G := K_m + lK_1$. Then $\mathrm{diff}(G) = 1$, $\kappa(G) = m \geq 4$, and $c(G) = 2m < 3m - 3 = \sigma_3(G) - 3$. Thus, conclusion (i) of Theorem 6 is best possible. Next, we let l, m be integers with $l \geq 4$ and $m \geq 2$, and let $G := 3K_1 + lK_m$. Then $\mathrm{diff}(G) = m > 1$, $\kappa(G) = 3$, and $c(G) = 3m + 3 = 3(m + 2) - 3 = \sigma_3(G) - 3$. Thus, conclusion (ii) of Theorem 6 is best possible. Finally, we let l, m be integers with $l \geq 3$ and $m \geq 3$, and let $G := 2K_1 + lK_m$. Then $\mathrm{diff}(G) = m > 1$, $\sigma_3(G) = 3(m + 1) = 3m + 3$, $\kappa(G) = 2$, and $c(G) = 2m + 2 < \sigma_3(G) - 3$. Since $p(G) = 3m + 2 = \sigma_3(G) - 1$, conclusion (iii) of Theorem 6 is also best possible.

2 Proof of Theorem 6

For standard graph-theoretic terminology not explained in this paper, we refer the reader to [4]. We denote by $N_G(x)$ the neighborhood of a vertex x in a graph G. For a subgraph H of G and a vertex $x \in V(G)$, we denote $N_H(x) := N_G(x) \cap V(H)$. Furthermore, for a subgraph H of G and $X \subseteq V(G)$, we write $N_H(X) := \bigcup_{x \in X} N_H(x) \setminus X$. If there is no confusion, we may identify a subgraph H of a graph G with its vertex set $V(H)$. We write a cycle (or a path) C with a given orientation by $\overrightarrow{C}$. Let C be a cycle or a path. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the successor and the predecessor of x on $\overrightarrow{C}$ by x^+ and x^-, respectively. For $X \subseteq V(C)$, we define $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$.

An *endblock* of a graph is a block that has at most one cut vertex. For convenience, we consider K_1 and K_2 as 2-connected graphs, and we call a 2-connected graph itself an endblock. For a block B, we write by $I(B)$ the set of vertices of B which are not cut vertices.

For $x, y \in V(G)$, let $D'_G(x, y) = \{|V(P)| : P \text{ is a longest } (x, y)\text{-path in } G\}$. For a 2-connected graph G, let $D'(G) = \min\{D'_G(x, y) : x, y \in V(G), x \neq y\}$. If G is connected and has cut vertices, we set $D'(G) = \max\{D'(B) : B \text{ is an endblock of } G\}$. For a trivial graph G, we define $D'(G) = 1$. (In fact, Fraisse and Jung [12] define an invariant $D(G)$ for a graph G. We can define $D'(G) = D(G) + 1$.)

Lemma 1 (Fraisse and Jung [12]) *Let G be a connected graph. Then there exist two vertices v_1, v_2 in G such that v_i is not a cut vertex of G and $D'(G) \geq d_G(v_i) + 1$ $(i = 1, 2)$. In particular, if $|V(G)| \geq 2$ then we can choose v_1 and v_2 such that $v_1 \neq v_2$.*

By the definition of $D'(G)$, we immediately obtain the following fact.

Fact 1 *Let G be a connected graph, and B be an endblock of G such that $D'(B) = D'(G)$. Let $u \in I(B)$ and $v \in V(G)$. Suppose that $u \neq v$ if $|V(G)| \geq 2$. Then there exists a (u, v)-path in G of order at least $D'(G)$.*

Proof of Theorem 6.

Suppose that G satisfies the assumption of Theorem 6 and $\mathrm{diff}(G) \geq 2$. Let Q be a longest path of G. Let C be a cycle and P_0 be a path with ends x and y_0 such that $V(C) \cup V(P_0) = V(Q)$, $V(C) \cap V(P_0) = \emptyset$ and $N_C(x) \neq \emptyset$. (Notice that there exist such a cycle C and a path P_0, because an endvertex of Q has at least two neighbors due to the 2-connectivity of G, and because Q is a longest path the neighbors must be in $V(Q)$.) Take such a cycle C and a path P_0 so that $|V(C)|$ is as large as possible. Note that $|P_0| \geq 2$ because $\mathrm{diff}(G) \geq 2$. We give an orientation to C and write an oriented cycle C by $\overrightarrow{C}$. A vertex $y \in V(P_0)$ is called *endable for* x if there exists an (x, y)-path P such that $V(P) = V(P_0)$. Let $L := \{y \in V(P_0) : y \text{ is endable for } x\}$ and let $L' := L \cup \{x\}$. We define $\mathcal{T} := \{(y, P) : y \in L \text{ and } P \text{ is an } (x, y)\text{-path such that } V(P) = V(P_0)\}$. For $(y, P) \in \mathcal{T}$, $\overrightarrow{P}$ is an oriented path from x to y, and $\overleftarrow{P}$ is an oriented path from y to x. Let $u_0 \in N_C(x)$. By the maximality of $|V(Q)|$ and $|V(C)|$, the following three claims hold. (We need the maximality of $|V(Q)|$ for Claim 2 and the case $u_2 \in N_C(G \setminus Q)$ of Claim 3, and the maximality of $|V(C)|$ for the case $u_2 \notin N_C(G \setminus Q)$ of Claim 3 and Claim 4.)

Claim 2 (i) *$N_{G\setminus Q}(L) = \emptyset$. Moreover, if $N_C(L) \neq \emptyset$ then $N_{G\setminus Q}(x) = \emptyset$.*

(ii) *If $u \in N_C(L')$, then $N_{G\setminus C}(u^+) = N_{G\setminus C}(u^-) = \emptyset$.*

Claim 3 *Suppose $u_1 \in N_C(L')$ and $u_2 \in N_C(G \setminus C)$ with $u_1 \neq u_2$. Let $C_1 = u_1^+ \overrightarrow{C} u_2$ and $C_2 = u_2^+ \overrightarrow{C} u_1$. Then the following statements hold.*

(i) *$N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+) = \emptyset$. In particular, $u_1^+ u_2^+ \notin E(G)$.*

(ii) *$N_{C_2}(u_1^+) \cap N_{C_2}(u_2^+)^- = \emptyset$.*

Claim 4 *Suppose $u_1 \in N_C(L)$ with $u_0 \neq u_1$. Let $C_0 = u_0^+ \overrightarrow{C} u_1$ and $C_1 = u_1^+ \overrightarrow{C} u_0$. Then $N_{C_0}(x)^+ \cap N_{C_0}(u_1^+)^- = \emptyset$ and $N_{C_1}(x)^+ \cap N_{C_1}(u_1^+)^- = \emptyset$.*

We divide the proof into two cases depending on $|N_C(L) \setminus \{u_0\}|$.

Case 1. $|N_C(L) \setminus \{u_0\}| \geq 2$.

Let $u_1, u_2 \in N_C(L) \setminus \{u_0\}$ with $u_1 \neq u_2$. Choose u_0 and u_1 so that $|V(u_0 \overrightarrow{C} u_1)|$ is as small as possible under the assumption of Case 1. Take $v \in (N_C(u_1^+) \cup N_C(u_2^+)) \cap V(u_0^+ \overrightarrow{C} u_1)$ so that $|V(u_0^+ \overrightarrow{C} v)|$ is as small as possible. Since $u_1 \in N_C(u_1^+) \cap V(u_0^+ \overrightarrow{C} u_1)$, there exists such a vertex v. By Claim 2 (ii), $N_{G\setminus C}(u_1^+) = N_{G\setminus C}(u_2^+) = \emptyset$. Therefore, by Claim 3 (i), $\{x, u_1^+, u_2^+\}$ is independent.

Let $D_0 = u_0^+ \overrightarrow{C} v^-$, $D_1 = u_1^+ \overrightarrow{C} u_2$ and $D_2 = u_2^+ \overrightarrow{C} u_0 \cup v \overrightarrow{C} u_1$. By the choice of u_0, u_1 and v, we have $N_{D_0}(x) = N_{D_0}(u_1^+) = N_{D_0}(u_2^+) = \emptyset$. Let h be an integer such that $v \in N_C(u_h^+)$, let $y \in N_{P_0}(u_h) \cap L$ and let P be a path with $(y, P) \in \mathcal{T}$. By the choice of C, the cycle $x u_0 \overleftarrow{C} u_h^+ v \overrightarrow{C} u_h y \overleftarrow{P} x$ is not longer than C, which implies $|V(D_0)| \geq |V(P)|$. By Claim 2 (i), we have $N_{G\setminus C}(x) = N_P(x)$. Since $N_P(x) \cup \{x\} \subseteq V(P)$, it follows that $|V(D_0)| \geq |N_{G\setminus C}(x)| + 1$. By Claim 3 (i) and (ii), we have $N_{D_1}(x)^+ \cap N_{D_1}(u_2^+) = \emptyset$ and $N_{D_1}(u_1^+)^- \cap N_{D_1}(u_2^+) = \emptyset$. By

Claim 4, $N_{D_1}(x)^+ \cap N_{D_1}(u_1^+)^- = \emptyset$. Hence $N_{D_1}(x)^+$, $N_{D_1}(u_1^+)^-$ and $N_{D_1}(u_2^+)$ are pairwise disjoint. Clearly, $N_{D_1}(x)^+ \cup N_{D_1}(u_1^+)^- \cup N_{D_1}(u_2^+) \subseteq V(D_1) \cup \{u_2^+\}$. Thus we obtain $|N_{D_1}(x)| + |N_{D_1}(u_1^+)| + |N_{D_1}(u_2^+)| \leq |V(D_1)| + 1$. Similarly, by Claim 3 (i) and (ii) and Claim 4, $N_{D_2}(x)^+$, $N_{D_2}(u_1^+)$ and $N_{D_2}(u_2^+)^-$ are pairwise disjoint. Since $N_{D_2}(x)^+ \cup N_{D_2}(u_1^+) \cup N_{D_2}(u_2^+)^- \subseteq V(D_2) \cup \{u_0^+, v^-, u_1^+\}$, we have $|N_{D_2}(x)| + |N_{D_2}(u_1^+)| + |N_{D_2}(u_2^+)| \leq |V(D_2)| + 3$. Thus we deduce

$$\begin{aligned}
c(G) &\geq |V(C)| \\
&= |V(D_0)| + |V(D_1)| + |V(D_2)| \\
&\geq |N_{G\setminus C}(x)| + 1 + |N_{D_1}(x)| + |N_{D_1}(u_1^+)| + |N_{D_1}(u_2^+)| - 1 \\
&\quad + |N_{D_2}(x)| + |N_{D_2}(u_1^+)| + |N_{D_2}(u_2^+)| - 3 \\
&\geq |N_G(x)| + |N_G(u_1^+)| + |N_G(u_2^+)| - 3 \\
&\geq \sigma_3(G) - 3.
\end{aligned}$$

Case 2. $|N_C(L) \setminus \{u_0\}| \leq 1$.

We first show that $p(G) \geq \sigma_3(G) - 1$. By the 2-connectedness of G, there exists a vertex $u_1 \in V(C) \setminus \{u_0\}$ such that $N_{G\setminus C}(u_1) \neq \emptyset$. Since $N_{G\setminus C}(u_0^-) = \emptyset$, we can choose u_1 so that $N_{G\setminus C}(u_1^+) = \emptyset$. By Claim 2 (ii) and by the choice of u_1, $N_{G\setminus C}(u_0^+) = N_{G\setminus C}(u_1^+) = \emptyset$. Hence, by Claim 3 (i), $\{y_0, u_0^+, u_1^+\}$ is independent.

Let $C_0 := u_0^+ \overrightarrow{C} u_1$ and $C_1 := u_1^+ \overrightarrow{C} u_0$. By Claim 3 (i) and (ii), we have $N_{C_0}(u_0^+)^- \cap N_{C_0}(u_1^+) = \emptyset$ and $N_{C_1}(u_0^+) \cap N_{C_1}(u_1^+)^- = \emptyset$. Clearly, $N_{C_0}(u_0^+)^- \cup N_{C_0}(u_1^+) \subseteq V(C_0)$ and $N_{C_1}(u_0^+) \cup N_{C_1}(u_1^+)^- \subseteq V(C_1)$. Thus we obtain $|N_{C_0}(u_0^+)| + |N_{C_0}(u_1^+)| \leq |V(C_0)|$ and $|N_{C_1}(u_0^+)| + |N_{C_1}(u_1^+)| \leq |V(C_1)|$. By Claim 2 (i), $N_{G\setminus C}(y_0) = N_P(y_0)$. Since $N_P(y_0) \cup \{y_0\} \subseteq V(P)$, it follows that $|V(P)| \geq |N_{G\setminus C}(y_0)| + 1$. By the assumption of Case 2, we have $|N_C(y_0)| \leq 2$. Hence we obtain

$$\begin{aligned}
p(G) &= |V(Q)| \\
&= |V(C_0)| + |V(C_1)| + |V(P)| \\
&\geq |N_{C_0}(u_0^+)| + |N_{C_0}(u_1^+)| + |N_{C_1}(u_0^+)| + |N_{C_1}(u_1^+)| + |N_{G\setminus C}(y_0)| + 1 \\
&\geq |N_G(u_0^+)| + |N_G(u_1^+)| + |N_G(y_0)| - 2 + 1 \\
&\geq \sigma_3(G) - 1.
\end{aligned}$$

If $\kappa(G) = 2$ then the conclusion (iii) holds. Henceforth we may assume that G is 3-connected.

Case 2.1. $|N_C(L) \setminus \{u_0\}| = 1$.

Let $N_C(L) \setminus \{u_0\} = \{u_1\}$ and $y \in N_{P_0}(u_1) \cap L$, and let P be a path with $(y, P) \in \mathcal{T}$. By the symmetry of x and y, we may assume that $N_C(x) \subseteq \{u_0, u_1\}$.

If $N_P(x)^- \cap N_P(y) \neq \emptyset$ then there exists a vertex $z \in L$ with $z^+ \notin L$. Assume not, and let $w \in N_P(x)^- \cap N_P(y)$. Then $P' = xw^+\overrightarrow{P}yw\overleftarrow{P}x^+$ is a path with $V(P') = V(P)$, and so $x^+ \in L$. This implies that $V(x^+\overrightarrow{P}y) \subseteq L$. Since G is 3-connected, it follows from Claim 2 (i) that $N_C(L) \setminus \{u_0, u_1\} \neq \emptyset$, which contradicts the assumption of Case 2. If $N_P(x)^- \cap N_P(y) \neq \emptyset$ then we take $z \in L$ so that $z^+ \notin L$; otherwise let $z := x$. In either case, note that $zy \notin E(G)$. By Claim 2 (ii), $\{z, y, u_1^+\}$ is independent.

By Claim 2 (i), $N_{G \setminus C}(z) = N_P(z)$ and $N_{G \setminus C}(y) = N_P(y)$. First assume $z = x$. Then $N_P(z)^- \cap N_P(y) = \emptyset$. Since $N_P(z)^- \cup N_P(y) \subseteq V(P) \setminus \{y\}$, we have $|V(P)| \geq |N_{G \setminus C}(z)| + |N_{G \setminus C}(y)| + 1$. Next assume $z \neq x$, and let $P_1 = x\overrightarrow{P}z$ and $P_2 = z^+\overrightarrow{P}y$. Since $z^+ \notin L$, it follows that $N_{P_1}(z)^+ \cap N_{P_1}(y) = \emptyset$ and $N_{P_2}(z) \cap N_{P_2}(y)^+ = \emptyset$. Clearly, $N_{P_1}(z)^+ \cup N_{P_1}(y) \subseteq V(P_1)$ and $N_{P_2}(z) \cup N_{P_2}(y)^+ \subseteq V(P_2)$. Thus, in either case, we obtain $|V(P)| \geq |N_{G \setminus C}(z)| + |N_{G \setminus C}(y)|$.

Choose $v \in N_C(u_1^+) \cap V(u_0^+\overrightarrow{C}u_1)$ so that $|V(u_0^+\overrightarrow{C}v)|$ is as small as possible. Let $D_0 := u_0^+\overrightarrow{C}v^-$ and $D_1 := u_1^+\overrightarrow{C}u_0 \cup v\overrightarrow{C}u_1$. By the choice of v and by Claim 2 (ii), $N_{D_0}(u_1^+) \cup N_{G \setminus C}(u_1^+) = \emptyset$. The choice of C implies $|V(D_0)| \geq |V(P)| \geq |N_{G \setminus C}(z)| + |N_{G \setminus C}(y)|$. Since $N_{D_1}(u_1^+) \subseteq V(D_1) \setminus \{u_1^+\}$, we have $|N_{D_1}(u_1^+)| \leq |V(D_1)| - 1$. By the assumption of Case 2, $N_C(z) \subseteq \{u_0, u_1\}$ and $N_C(y) \subseteq \{u_0, u_1\}$. Therefore we obtain

$$\begin{aligned} c(G) &\geq |V(C)| \\ &= |V(D_0)| + |V(D_1)| \\ &\geq |N_{G \setminus C}(z)| + |N_{G \setminus C}(y)| + |N_{D_1}(u_1^+)| + 1 \\ &\geq (|N_G(z)| - 2) + (|N_G(y)| - 2) + |N_G(u_1^+)| + 1 \\ &\geq \sigma_3(G) - 3. \end{aligned}$$

Case 2.2. $|N_C(L) \setminus \{u_0\}| = 0$.

Let $y \in L$, and let P be a path with $(y, P) \in \mathcal{T}$. Since $G \setminus \{u_0\}$ is 2-connected and $\min\{|V(P)|, |V(C) \setminus \{u_0\}|\} \geq 2$, there exist two vertex disjoint paths R_i $(i = 1, 2)$ such that R_i connects z_i and u_i, where $\{z_i\} = V(R_i) \cap V(P)$ and $\{u_i\} = V(R_i) \cap (V(C) \setminus \{u_0\})$. By the assumption of Case 2.2 and by Claim 2 (i), we have $z_i \neq y$. Choose such a path R_1 so that $|V(z_1\overrightarrow{P}y)|$ is as small as possible.

By considering the reverse orientation of C if necessary, we may assume that $u_2 \in V(u_1^+\overrightarrow{C}u_0^-)$.

First, we show the existence of a long path between z_1 and x or z_2. Since $z_1 \neq y$ by the assumption of Case 2.2, there exists a vertex z_1^+. Let $P_1 := x\overrightarrow{P}z_1$ and $P_2 := z_1^+\overrightarrow{P}y$. Note that $|V(P_1)| \geq 2$ by the choice of R_1. Let H_1 be a component of $G \setminus (C \cup P_1)$ such that $V(P_2) \subseteq V(H_1)$ and let B_1 be an endblock of H_1 such that $D'(B_1) = D'(H_1)$. Note that $V(R_i) \cap V(H_1) = \emptyset$ for $i = 1, 2$ by the choice of R_1. By Lemma 1, there exist vertices $v_1, v_2 \in V(B_1)$ such that $D'(B_1) \geq d_{H_1}(v_i) + 1$ $(i = 1, 2)$.

Claim 5 *Let $z \in V(x\overrightarrow{P}z_1^-)$. For some i, there exists a (z, z_1)-path in $P_1 \cup H_1$ of order at least $|N_G(v_i)|$.*

Proof. By the choice of R_1, we have $N_G(H_1) \subseteq V(P_1) \cup \{u_0\}$. Since $G \setminus \{u_0\}$ is 2-connected, there exist two edges $e_1 := a_1b_1$ and $e_2 := a_2b_2$ with $a_1, a_2 \in V(P_1)$ and $a_1 \neq a_2$ such that one connects $I(B_1)$ and $V(P_1)$ and the other connects $V(H_1)$ and $V(P_1)$. In particular, if $|V(H_1)| \geq 2$ then we can choose b_1 and b_2 such that $b_1 \neq b_2$. Since $z_1^+z_1$ is an edge connecting $V(H_1)$ and $V(P_1)$, we can choose such two edges so that $a_1 \in V(z^+\overrightarrow{P}z_1)$. Choose e_1 and e_2 so that (i) $|V(z^+\overrightarrow{P}a_1)|$ is as small as possible and (ii) $|V(x\overrightarrow{P}a_2)|$ is as small as possible, subject to (i). If $|V(H_1)| \geq 2$, then we may assume that $v_1 \neq b_1$. Therefore it follows from the choice of e_1 and e_2 that $N_{P_1}(v_1) \cap V(z^+\overrightarrow{P}a_1^-) = \emptyset$ and $N_{P_1}(v_1) \cap V(x\overrightarrow{P}a_2^-) \subseteq \{a_1\}$.

By Fact 1, there exists a (b_1, b_2)-path T_1 in H_1 with $|V(T_1)| \geq D'(B_1) \geq |N_{H_1}(v_1)| + 1$. Let

$$Q_1 := \begin{cases} z\overrightarrow{P}a_1b_1T_1b_2a_2\overrightarrow{P}z_1 & \text{if } a_2 \in V(a_1^+\overrightarrow{P}z_1), \\ z\overleftarrow{P}a_2b_2T_1b_1a_1\overrightarrow{P}z_1 & \text{otherwise.} \quad \text{(See Figure 1.)} \end{cases}$$

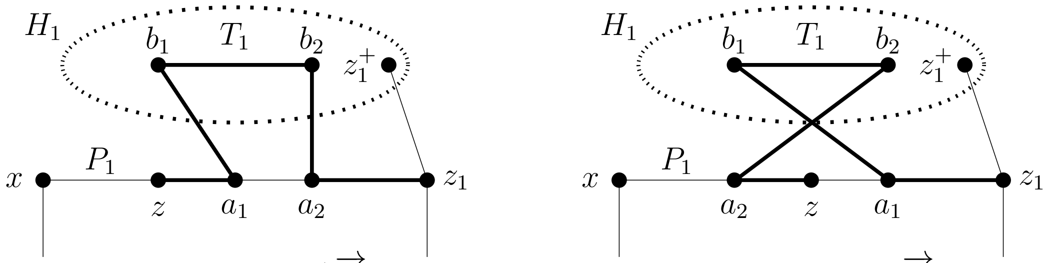

Figure 1: The path Q_1.

Then $N_{P_1}(v_1) \subseteq V(P_1) \cap V(Q_1)$ and hence $|N_{P_1}(v_1)| \leq \big|V(P_1) \cap V(Q_1)\big|$. Because $N_{G \setminus P_1}(v_1) = N_{H_1}(v_1) \cup N_C(v_1)$ and $N_C(v_1) \subseteq \{u_0\}$, we have $|N_{G\setminus P_1}(v_1)| \leq |N_{H_1}(v_1)| + 1$. Therefore we obtain

$$\begin{aligned} |V(Q_1)| &= |V(P_1) \cap V(Q_1)| + |V(T_1)| \\ &\geq |N_{P_1}(v_1)| + |N_{H_1}(v_1)| + 1 \\ &\geq |N_G(v_1)|, \end{aligned}$$

and so Q_1 is a (z, z_1)-path of order at least $|N_G(v_1)|$. $\square$

By Claim 5, for some $i_x, i_z \in \{1, 2\}$ there exist an (x, z_1)-path P_x and a (z_2, z_1)-

path P_z in H of order at least $|N_G(v_{i_x})|$ and $|N_G(v_{i_z})|$, respectively. Hereafter we never consider P_x and P_z at the same time, and by symmetry between v_1 and v_2, we may assume that $i_x = i_z = 1$.

Next, we will prove the existence of a cycle of length at least $\sigma_3(G) - 2$. To prove it, we focus on $u_2^+ \overrightarrow{C} u_0^-$.

Claim 6 *Suppose that there exists a vertex $w^* \in V(u_2^+ \overrightarrow{C} u_0^-)$ such that for every $w \in V(u_2^+ \overrightarrow{C} w^*)$, $N_{G \setminus C}(w) \neq \emptyset$ or $N_C(w) \cap V(u_2^+ \overrightarrow{C} w) \setminus \{w, w^-\} \neq \emptyset$. Then there exist a vertex $z \in V(G \setminus C)$ and a (z, u_2)-path R such that $V(R) = V(u_2 \overrightarrow{C} w^*) \cup \{z\}$.*

Proof. If $w^* = u_2^+$, then we can easily show the existence of a desired path because $N_{G \setminus C}(w^*) \neq \emptyset$. Thus, we may assume that $w^* \neq u_2^+$. We define a function $f : V(u_2^+ \overrightarrow{C} w^*) \to V(u_2^+ \overrightarrow{C} w^*)$ as follows. For $w \in V(u_2^+ \overrightarrow{C} w^*)$, if $N_{G \setminus C}(w) \neq \emptyset$ then let $f(w) = w$; otherwise let $u \in N_C(w) \cap V(u_2^+ \overrightarrow{C} w) \setminus \{w, w^-\}$ and $f(w) = u^+$. Moreover, we define $w_0 = w^*$ and $w_{i+1} = f(w_i)$ for $i = 0, 1, 2, \cdots$. Since $w_{i+1} \in V(u_2^+ \overrightarrow{C} w_i)$ for any i, there exists an integer j such that $w_{j+1} = w_j$. Take such j as small as possible. Note that $w_j \neq u_2^+$ because $w_j^- \in V(u_2^+ \overrightarrow{C} w^*)$ unless $w^* = w_0 = u_2^+$. Let $z \in N_{G \setminus C}(w_j)$ and let

$$R := \begin{cases} z w_j \overrightarrow{C} w_{j-1}^- w_{j-2} \overrightarrow{C} w_{j-3}^- \cdots w_2^- w_1 \overrightarrow{C} w^* w_1^- \overleftarrow{C} w_2 \cdots w_j^- \overleftarrow{C} u_2 & \text{if } j \text{ is odd}, \\ z w_j \overrightarrow{C} w_{j-1}^- w_{j-2} \overrightarrow{C} w_{j-3}^- \cdots w_2 \overrightarrow{C} w_1^- w^* \overleftarrow{C} w_1 w_2^- \cdots w_j^- \overleftarrow{C} u_2 & \text{if } j \text{ is even}. \end{cases}$$

See Figure 2. Then R is a desired path. $\square$

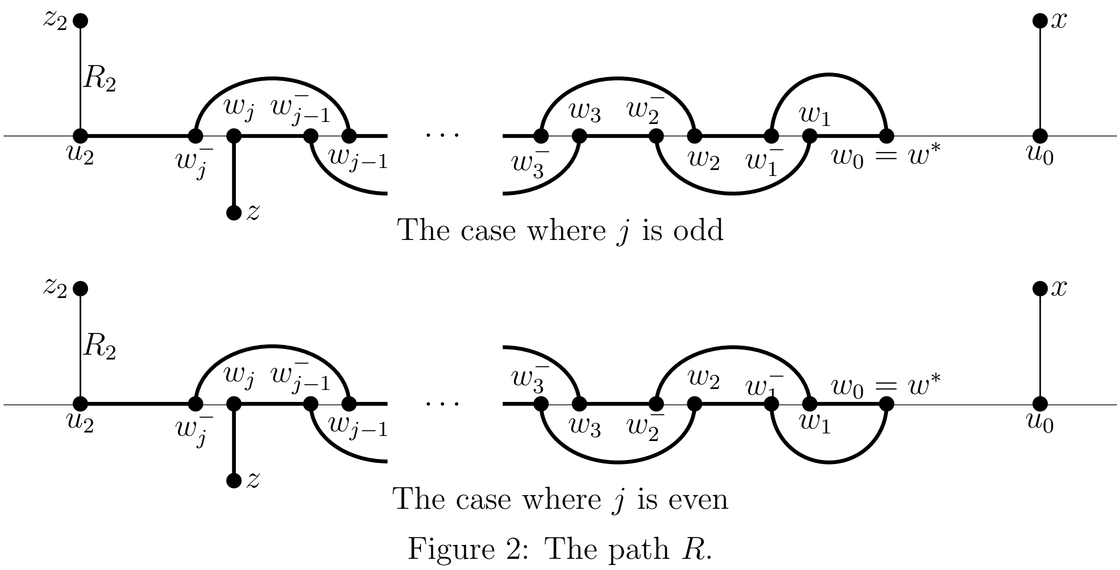

Figure 2: The path R.

We show that there exists a vertex $w_0 \in V(u_2^+ \overrightarrow{C} u_0^-)$ such that $N_{G \setminus C}(w_0) = \emptyset$ and $N_C(w_0) \cap V(u_2^+ \overrightarrow{C} w_0) \setminus \{w_0, w_0^-\} = \emptyset$. Assume not. By applying Claim 6 as $w^* = u_0^-$, there exists a (z, u_2)-path R such that $z \in V(G \setminus C)$ and $V(R) = V(u_2 \overrightarrow{C} u_0^-) \cup \{z\}$. Then $zRu_2 \overleftarrow{C} u_0 x \overrightarrow{P} y$ contradicts the choice of Q or C. Choose such a vertex w_0 so that $|V(u_2^+ \overrightarrow{C} w_0)|$ is as small as possible.

Choose $v \in (N_C(u_0^+) \cup N_C(w_0) \cup \{u_2\}) \cap V(u_1^+ \overrightarrow{C} u_2)$ so that $|V(u_1^+ \overrightarrow{C} v)|$ is as small as possible. Let

$$C' := \begin{cases} u_1 R_1 z_1 P_z z_2 R_2 u_2 \overrightarrow{C} u_1 & \text{if } v = u_2, \\ u_0^+ \overrightarrow{C} u_1 R_1 z_1 P_x x u_0 \overleftarrow{C} v u_0^+ & \text{if } v \in N_C(u_0^+), \\ w_0 \overrightarrow{C} u_1 R_1 z_1 P_z z_2 R_2 u_2 \overleftarrow{C} v w_0 & \text{if } v \in N_C(w_0). \end{cases}$$

Let $D_0 = u_0^+ \overrightarrow{C} u_1 \cup v \overrightarrow{C} u_2$, $D_1 = u_1^+ \overrightarrow{C} v^-$, $D_2 = u_2^+ \overrightarrow{C} w_0^-$ and $D_3 = w_0 \overrightarrow{C} u_0$. Note that $V(C) = V(D_0 \cup D_3 \cup D_1 \cup D_2)$ and $V(C' \cap C) \subseteq V(D_0 \cup D_3)$.

We prove $N_{D_0}(u_0^+)^- \cap N_{D_0}(w_0) = \emptyset$. Assume not, say $w \in N_{D_0}(u_0^+)^- \cap N_{D_0}(w_0)$. Then $w_0 \neq u_2^+$ by Claim 3. By applying Claim 6 as $w^* = w_0^-$, there exist a vertex $z \in V(G \setminus C)$ and a (z, u_2)-path R such that $V(R) = V(u_2 \overrightarrow{C} w_0^-) \cup \{z\}$. Then $zRu_2 \overleftarrow{C} w^+ u_0^+ \overrightarrow{C} w w_0 \overrightarrow{C} u_0 x \overrightarrow{P} y$ contradicts the choice of Q or C. Therefore $N_{D_0}(u_0^+)^- \cap N_{D_0}(w_0) = \emptyset$, and in particular, $u_0^+ w_0 \notin E(G)$. We can similarly prove $N_{D_3}(u_0^+) \cap N_{D_3}(w_0)^- = \emptyset$. Clearly, $N_{D_0}(u_0^+)^- \cup N_{D_0}(w_0) \subseteq V(D_0) \cup \{v^-\}$ and $N_{D_3}(u_0^+) \cup N_{D_3}(w_0)^- \subseteq V(D_3)$.

By the choice of w_0, $N_{G \setminus C}(w_0) = \emptyset$. By Claim 2 (ii), we have $N_{G \setminus C}(u_0^+) = \emptyset$. Therefore $\{v_1, u_0^+, w_0\}$ is independent.

By the choice of v, $N_{D_1}(u_0^+) = N_{D_1}(w_0) = \emptyset$. By the choice of w_0, $N_{D_2}(w_0) \subseteq \{w_0^-\}$. We show that $N_{D_2}(u_0^+) = \emptyset$. Assume not, say $w \in N_{D_2}(u_0^+)$. By Claim 3, we obtain $w \neq u_2^+$. By applying Claim 6 as $w^* = w^-$, there exist a vertex $z \in V(G \setminus C)$ and a (z, u_2)-path R such that $V(R) = V(u_2 \overrightarrow{C} w^-) \cup \{z\}$. Then $zRu_2 \overleftarrow{C} u_0^+ w \overrightarrow{C} u_0 x \overrightarrow{P} y$ contradicts the choice of Q or C. Thus we deduce

$$\begin{aligned} |V(C' \cap C)| &\geq |V(D_0)| + |V(D_3)| \\ &\geq |N_{D_0}(u_0^+)| + |N_{D_0}(w_0)| - 1 + |N_{D_3}(u_0^+)| + |N_{D_3}(w_0)| \\ &\geq |N_G(u_0^+)| + |N_G(w_0)| - 2. \end{aligned}$$

Since $V(P_x) \subseteq V(C' \setminus C)$ or $V(P_z) \subseteq V(C' \setminus C)$, it follows that $|V(C' \setminus C)| \geq |N_G(v_1)|$. Therefore, we obtain $c(G) \geq |V(C')| \geq |N_G(u_0^+)| + |N_G(w_0)| + |N_G(v_1)| - 2 \geq \sigma_3(G) - 2$. $\square$

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