A Note on Total Excess of Spanning Trees

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Abstract

A graph G is said to be t-tough if $|S| \geq t \cdot \omega(G-S)$ for any subset S of $V(G)$ with $\omega(G-S) \geq 2$, where $\omega(G-S)$ is the number of components in $G-S$. Win proved that for any integer $n \geq 3$ every $\frac{1}{n-2}$-tough graph has a spanning tree with maximum degree at most n. In this paper, we investigate t-tough graphs including the cases where $t \notin \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, and consider spanning trees in such graphs. Using the notion of total excess, we prove that if G is $\frac{1-\varepsilon}{n-2+\varepsilon}$-tough for an integer $n \geq 2$ and a real number ε with $\frac{2}{|V(G)|} \leq \varepsilon \leq 1$, then G has a spanning tree T such that

$$\sum_{v \in V(G)} \max\{0, \deg_T(v) - n\} \leq \varepsilon |V(G)| - 2.$$

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We also investigate the relation between spanning trees in a graph obtained by different pairs of parameters (n, ε). As a consequence, we prove the existence of "a universal tree" in a connected t-tough graph G, that is a spanning tree T such that $\sum_{v \in V(T)} \max\{0, \deg_T(v) - n\} \leq \varepsilon|V(G)| - 2$ for any integer $n \geq 2$ and real number ε with $\frac{2}{|V(G)|} \leq \varepsilon \leq 1$, which satisfy $t \geq \frac{1-\varepsilon}{n-2+\varepsilon}$.

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1 Introduction

In this paper, we consider finite undirected graphs without multiple edges or loops. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. We denote by $\deg_G(v)$ the degree of a vertex v in G.

For $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with their incident edges. We denote by $\omega(G)$ the number of components of G. A graph G is said to be *t-tough* if $|S| \geq t \cdot \omega(G - S)$ for any subset S of $V(G)$ with $\omega(G - S) \geq 2$. The largest real number t such that G is t-tough is called the *toughness* of G, and is denoted by $t(G)$. If G is a complete graph, its toughness is defined to be ∞.

An n-tree is a spanning tree whose maximum degree is less than or equal to n. Win [5] gave a sufficient condition for a graph to contain an n-tree, in terms of toughness.

Theorem 1 ([5]) *Let n be an integer with $n \geq 2$. Suppose that G is a connected graph satisfying the following condition:*

For every nonempty subset S of $V(G)$, $\omega(G - S) \leq (n - 2)|S| + 2$.

Then, G has an n-tree.

More results on toughness and factors can be found in [1] and [2].

As a corollary to Theorem 1, we can easily see that every $\frac{1}{n-2}$-tough graph has an n-tree for an integer $n \geq 3$. In this paper, we consider the graphs with toughness of intermediate fractions, other than $1, \frac{1}{2}, \frac{1}{3}, \ldots$, and discuss the spanning trees contained in such graphs.

We introduce the notion of total excess. Let G be a graph. Let n be an integer. For a spanning subgraph H of G, we define the *n-excess* of a vertex v as $\max\{0, \deg_H(v) - n\}$. We define the *total n-excess* $\mathrm{te}(H, n)$ to be the summation of n-excess of all vertices, namely,

$$\mathrm{te}(H, n) = \sum_{v \in V(H)} \max\{0, \deg_H(v) - n\}.$$

The following theorem gives a sufficient condition for a graph to have a spanning tree with bounded total excess.

Theorem 2 ([3]) *Suppose that $n \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition;*

For any subset S of $V(G)$, $\omega(G - S) \leq (n-2)|S| + b + 2$.

Then, G has a spanning tree T with $\mathrm{te}(T, n) \leq b$.

Using this theorem, we can easily prove the following corollary.

Corollary 1 *Let G be a connected graph, $n \geq 2$ be an integer and ε be a real number with $\frac{2}{|V(G)|} \leq \varepsilon \leq 1$. If G is $\frac{1-\varepsilon}{n-2+\varepsilon}$-tough, then there exists a spanning tree T such that*

$$\mathrm{te}(T, n) \leq \varepsilon |V(G)| - 2.$$

Proof of Corollary 1. Let S be a nonempty subset of $V(G)$. If $\omega(G-S) \geq 2$, then since G is $\frac{1-\varepsilon}{n-2+\varepsilon}$-tough, we obtain

$$(n - 2 + \varepsilon)|S| \geq (1 - \varepsilon)\omega(G - S),$$

or

$$\begin{aligned}\omega(G-S) &\le (n-2)|S| + \varepsilon(|S| + \omega(G-S)) \\ &\le (n-2)|S| + (\varepsilon|V(G)| - 2) + 2.\end{aligned}$$

The last inequality holds even when $\omega(G-S) = 1$. Thus, it follows from Theorem 2 that there exists a spanning tree T with $\mathrm{te}(T, n) \le \varepsilon|V(G)| - 2$. $\square$

For a given graph G, there are many pairs (n, ε) which satisfy the assumption of Corollary 1. Therefore, we obtain a lot of spanning trees from such pairs by applying Corollary 1. Needless to say, they are not necessarily the same tree. But sometimes, one spanning tree may satisfy the conclusion of Corollary 1 for many distinct pairs (n, ε). In the next section, we discuss the relation of the conclusions of Corollary 1 for distinct pairs (n, ε).

2 Relation Between the Spanning Trees

We obtained a lot of spanning trees by applying Corollary 1. In this section, we compare these spanning trees.

Formally, for an integer $n \ge 2$ and for positive real numbers $\varepsilon_1 \le 1$ and $\varepsilon_2 \le 1$, we set

$$\frac{1-\varepsilon_1}{n-2+\varepsilon_1} = \frac{1-\varepsilon_2}{(n+1)-2+\varepsilon_2} \tag{1}$$

and suppose that G is a connected graph satisfying $\frac{|S|}{\omega(G-S)} \ge \frac{1-\varepsilon_1}{n-2+\varepsilon_1} = \frac{1-\varepsilon_2}{(n+1)-2+\varepsilon_2}$ for any nonempty subset S of $V(G)$. And suppose $\varepsilon_2 \ge \frac{2}{|V(G)|}$ (which implies $\varepsilon_1 \ge \frac{2}{|V(G)|}$ since $\varepsilon_1 = \varepsilon_2 + \frac{1-\varepsilon_2}{n} \ge \varepsilon_2$). Note that by (1), we get

$$\varepsilon_2 = \frac{n\varepsilon_1 - 1}{n-1}.$$

By applying Corollary 1 to the pairs (n, ε_1) and $(n+1, \varepsilon_2)$, we obtain two spanning trees T_1 and T_2 with $\mathrm{te}(T_1, n) \le \varepsilon_1|V(G)| - 2$ and $\mathrm{te}(T_2, n+1) \le$

$\varepsilon_2|V(G)|-2 = \frac{n\varepsilon_1-1}{n-1}|V(G)|-2$, respectively. We shall show that T_2 can play the same role as T_1.

Let $V_k(T_2) = \{v \in V(G) \,|\, \deg_{T_2}(v) = k\}$. We shall estimate $\mathrm{te}(T_2, n)$. Since $\mathrm{te}(T_2, n+1) \leq \varepsilon_2|V(G)|-2$, we have

$$\frac{n\varepsilon_1 - 1}{n-1}|V(G)| - 2 \geq \sum_{l\geq 1}(l-1)|V_{n+l}(T_2)|. \qquad (2)$$

On the other hand, since $|E(T_2)| = |V(G)|-1$ and $2|E(T_2)| = \sum_{k\geq 1} k|V_k(T_2)|$, we have

$$2|V(G)| - 2 = \sum_{k\geq 1} k|V_k(T_2)|,$$

and hence

$$|V(G)| - 2 = \sum_{k\geq 1}(k-1)|V_k(T_2)| \geq \sum_{l\geq 1}(n+l-1)|V_{n+l}(T_2)|. \qquad (3)$$

By computing (2) $\times \frac{n-1}{n}$ + (3) $\times \frac{1}{n}$, we deduce

$$\varepsilon_1|V(G)| - 2 \geq \sum_{l\geq 1} l|V_{n+l}(T_2)| = \mathrm{te}(T_2, n).$$

Thus, T_2 has the same bound on the total n-excess as T_1.

Applying the above argument repeatedly, we obtain the following theorem, in which a spanning tree T of G is said to be *good at* a pair (n, ε), if T satisfies the conclusion of Corollary 1, namely $\mathrm{te}(T, n) \leq \varepsilon|V(G)| - 2$.

Theorem 3 *Let $\frac{2}{|V(G)|} \leq \varepsilon_0 \leq 1$ and $n_0 \geq 2$. If a spanning tree T of G is good at (n_0, ε_0), then T is also good at all pairs (n, ε) such that $2 \leq n \leq n_0$ and $\frac{1-\varepsilon}{n-2+\varepsilon} = \frac{1-\varepsilon_0}{n_0-2+\varepsilon_0}$.*

3 A Universal Tree

In this section, we shall prove the existence of a universal tree, that is a spanning tree which is good at any pair (n, ε) satisfying the assumption of Corollary 1.

Theorem 4 *Let G be a connected graph and let $t = t(G)$. Then there is a spanning tree T of G such that* $\mathrm{te}(T, n) \le \varepsilon|V(G)| - 2$ *for any integer $n \ge 2$ and real number $\frac{2}{|V(G)|} \le \varepsilon \le 1$, which satisfy $t \ge \frac{1-\varepsilon}{n-2+\varepsilon}$.*

Proof of Theorem 4. Consider all pairs (n, ε) satisfying $\frac{1-\varepsilon}{n-2+\varepsilon} = t$. Among them, let $n \ge 2$ be the maximum integer such that the corresponding ε satisfies $\varepsilon \ge \frac{2}{|V(G)|}$, equivalently $\frac{1-t(n-2)}{1+t} \ge \frac{2}{|V(G)|}$.

Claim 1. G has an $(n+1)$-tree.

Proof of Claim 1. Let ε' be the real number corresponding to $n+1$, namely $\frac{1-\varepsilon'}{(n+1)-2+\varepsilon'} = t$. By the definition of n, we have $\varepsilon' < \frac{2}{|V(G)|}$. Let $\varepsilon_0 = \frac{2}{|V(G)|}$ so that $\varepsilon_0 > \varepsilon'$. Then,

$$t = \frac{1-\varepsilon'}{(n+1)-2+\varepsilon'} > \frac{1-\varepsilon_0}{(n+1)-2+\varepsilon_0},$$

and hence by Corollary 1, G has a spanning tree T such that $\mathrm{te}(T, n+1) \le \varepsilon_0|V(G)| - 2 = 0$. Thus, T is an $(n+1)$-tree.

Let T be an $(n+1)$-tree of G such that $|V_{n+1}(T)|$ is smallest possible.

Claim 2. $\mathrm{te}(T, n) \le \varepsilon|V(G)| - 2$, where ε is the real number satisfying $\frac{1-\varepsilon}{n-2+\varepsilon} = t$.

We first finish the proof of Theorem 4 by using Claim 2. We shall prove that T is a desired spanning tree of G, that is T is good at any pair (n', ε') such that n' is an integer at least 2 and $\frac{2}{|V(G)|} \le \varepsilon' \le 1$ satisfying $t \ge \frac{1-\varepsilon'}{n'-2+\varepsilon'}$.

Suppose that $2 \le n' \le n$. Let ε' be a real number satisfying $\frac{2}{|V(G)|} \le \varepsilon' \le 1$ and $t \ge \frac{1-\varepsilon'}{n'-2+\varepsilon'}$, namely $\max\{\frac{1-t(n'-2)}{1+t}, \frac{2}{|V(G)|}\} \le \varepsilon' \le 1$. Since the value $\varepsilon'|V(G)| - 2$ is monotonically increasing with ε', it suffices to prove that T is good at (n', ε') for $\varepsilon' = \frac{1-t(n'-2)}{1+t}$, i.e., $\frac{1-\varepsilon'}{n'-2+\varepsilon'} = t$. If $n' = n$, then the assertion is equivalent to Claim 2. Moreover, by using Theorem 3, we can verify that T is good at any pair (n', ε') with $2 \le n' \le n$ and $\frac{1-\varepsilon'}{n'-2+\varepsilon'} = t$.

Suppose that $n' \geq n+1$. Since T is an $(n+1)$-tree, we have $\mathrm{te}(T, n') = 0$, and hence T is good at $(n', \frac{2}{|V(G)|})$. We can easily verify that T is good at any pair (n', ε') with $n' \geq n+1$ and $\frac{2}{|V(G)|} \leq \varepsilon' \leq 1$.

Thus, T is good at any pair (n', ε') satisfying $n' \geq 2$, $\frac{2}{|V(G)|} \leq \varepsilon' \leq 1$ and $t \geq \frac{1-\varepsilon'}{n'-2+\varepsilon'}$.

In the rest of this paper, we shall prove Claim 2. In order to prove Claim 2, we introduce the notion of a bridge. For $S \subseteq V(G)$, an *S-bridge* of G is a subgraph consisting of a component C of $G - S$ together with the edges joining S and C, or an edge both of whose ends are contained in S.

Proof of Claim 2. For $S \subseteq V(G)$, let $\mathcal{T}(S)$ denote the set of $(n+1)$-trees T' of G such that $V_{n+1}(T') = V_{n+1}(T)$ and the vertex sets of the S-bridges of T' coincide with those of the S-bridges of T. Let $A_0 = V_{n+1}(T)$. Note that $\mathrm{te}(T, n) = |A_0|$ since T is an $(n+1)$-tree. If $A_0 = \emptyset$, then $\mathrm{te}(T, n) = 0$, which means T is a desired tree. Thus, we may assume $A_0 \neq \emptyset$.

Let $A_1 = A_0 \cup \{x \in V(G) | \deg_{T'}(x) = n \text{ for all } T' \in \mathcal{T}(A_0)\}$.

Subclaim 1. Each edge of G which connects different components of $T - A_0$ has an end vertex in A_1.

Proof of Subclaim 1. Let $uv \in E(G)$ be an edge which connects different components of $T - A_0$. Then, for every $T' \in \mathcal{T}(A_0)$, u and v are contained in different components of $T' - A_0$. Suppose $u \notin A_1$ and $v \notin A_1$. Then, there exist $T_1, T_2 \in \mathcal{T}(A_0)$ satisfying $\deg_{T_1}(u) < n$ and $\deg_{T_2}(v) < n$. By replacing the A_0-bridge in T_1 that contains v with the A_0-bridge in T_2 that contains v, we get another $(n+1)$-tree $T_3 \in \mathcal{T}(A_0)$ such that the degrees of u and v are less than n.

There exists a (u, v)-path $T_3(u, v)$ in T_3, and the path contains a vertex w of A_0. By adding the edge uv, and removing one of the edges in $T_3(u, v)$ incident with w, we obtain an $(n+1)$-tree T_3' such that $V_{n+1}(T_3') \subseteq V_{n+1}(T) \setminus \{w\}$ since the degree of w is less than $n+1$. This contradicts the minimality of $|V_{n+1}(T)|$. Therefore, we establish $u \in A_1$ or $v \in A_1$.

To continue this inductively, we define $A_{j+1} = A_j \cup \{x \in V(G) | \deg_{T'}(x) = n$ for all $T' \in \mathcal{T}(A_j)\}$. Then we can show the following subclaim by the same argument as Subclaim 1.

Subclaim 2. Each edge connecting different components of $T - A_j$ has an end vertex in A_{j+1}.

Proof of Subclaim 2. Let $uv \in E(G)$ be an edge which connects different components of $T - A_j$. Then, for every $T' \in \mathcal{T}(A_j)$, u and v are contained in different components of $T' - A_j$. Suppose $u \notin A_{j+1}$ and $v \notin A_{j+1}$. Then, there exist $T_1, T_2 \in \mathcal{T}(A_j)$ satisfying $\deg_{T_1}(u) < n$ and $\deg_{T_2}(v) < n$. By replacing the A_j-bridge in T_1 that contains v with the A_j-bridge in T_2 that contains v, we get another $(n+1)$-tree $T_3 \in \mathcal{T}(A_j)$ such that the degrees of u and v are less than n.

There exists a (u, v)-path $T_3(u, v)$ in T_3, and the path contains a vertex w of A_j. If $w \in A_{j-1}$, then uv is an edge connecting different components of $T - A_{j-1}$. By the induction hypothesis of Subclaim 2 (or by Subclaim 1), $u \in A_j$ or $v \in A_j$. This contradicts the fact that u and v are in $T - A_j$. Thus, $w \in A_j \setminus A_{j-1}$. Furthermore, $T_3(u, v)$ does not contain any vertex $w' \in A_{j-1}$. Hence, u and v are in the same component of $T - A_{j-1}$. By adding the edge uv, and removing one of the edges in $T_3(u, v)$ incident with w, we obtain an $(n+1)$-tree $T_3' \in \mathcal{T}(A_{j-1})$ such that the degree of w is less than n. This contradicts the fact $w \in A_j$. Therefore, we establish $u \in A_{j+1}$ or $v \in A_{j+1}$.

We get the following sequence of vertex sets, where $V_{\geq n}(T)$ is the set of vertices whose degree is at least n in T.

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subseteq \cdots \subseteq V_{\geq n}(T).$$

Because $V_{\geq n}(T)$ is a finite set, we get $A_m = A_{m+1}$ at some integer m. Then, by Subclaim 2, A_m has the property that any edge connecting different components of $T - A_m$ has an end vertex in A_m. In other words, there is no edge of G connecting different components of $T - A_m$. This implies that for $S = A_m$, we have $\omega(G - S) = \omega(T - S)$.

Recall that $\omega(T - S) \geq 2 + \sum_{s \in S}(\deg_T(s) - 2)$ for a tree T and nonempty subset $S \subseteq V(T)$. Let $B = S \setminus A_0$. Then,

$$\omega(G - S) = \omega(T - S) \geq 2 + (n-1)|A_0| + (n-2)|B|. \tag{4}$$

In particular, we have $\omega(G - S) \geq 2$ by (4). Since $\frac{|S|}{\omega(G-S)} \geq t = \frac{1-\varepsilon}{n-2+\varepsilon}$,

$$(1-\varepsilon)\omega(G - S) \leq (n - 2 + \varepsilon)|S|.$$

$$\begin{aligned} \omega(G - S) &\leq (n-2)|S| + \varepsilon(|S| + \omega(G - S)) \\ &\leq (n-2)(|A_0| + |B|) + \varepsilon|V(G)|. \end{aligned} \tag{5}$$

By (4) and (5), we have $\varepsilon|V(G)| - 2 \geq |A_0| = \mathrm{te}(T, n)$. This completes the proof of Claim 2. $\square$

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