Hamiltonian cycles in bipartite quadrangulations on the torus

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Abstract

In this paper, we shall prove that every bipartite quadrangulation G on the torus admits a simple closed curve visiting each face and each vertex of G exactly once but crossing no edge. As an application, we conclude that the radial graph of any bipartite quadrangulation on the torus has a hamiltonian cycle.

Keywords: hamiltonian cycle, quadrangulation, bipartite graph, torus

1 Introduction

We consider only finite *simple* graphs, that is, those without loops and multiple edges. A *k-cycle* means a cycle of length k. We say that a cycle (or a path) C of a graph G is *Hamiltonian* if C visits each vertex of G exactly once. We say that G is *Hamiltonian* if G has a Hamiltonian cycle.

A *surface* means a connected compact 2-dimensional manifold possibly with boundary. A simple closed curve l on a surface F^2 is said to be *essential* if l does not bound a 2-cell on F^2. Let G be a *map* on a surface, that is, a fixed embedding of a graph on the surface. Let $V(G)$, $E(G)$ and $F(G)$ denote the sets of the vertices, edges and faces of G, respectively. Let G be a map in a surface F^2 and let l be a simple closed curve on F^2 intersecting only vertices of G. *Cutting* G along l is to cut F^2 along l so that in the map on the resulting surface, each vertex of G lying on l appears twice on the boundary.

The following is one of the most celebrated theorems in graph theory.

THEOREM 1 (Tutte [6]) *Every 4-connected plane graph is Hamiltonian.*

Theorem 1 has been strengthened by Thomassen [10] to show that every 4-connected plane graph is *Hamiltonian-connected*, i.e., for any two vertices, there exists a Hamiltonian path joining them. (A Hamiltonian path between two adjacent vertices a, b can be regarded as a Hamiltonian cycle through the edge ab.)

For the projective plane, the following theorem is known.

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Theorem 2 (Thomas and Yu [7]) *Every 4-connected projective plane map is Hamiltonian.*

Note that in Theorems 1 and 2, the 4-connectedness cannot be omitted, since some non-4-connected graphs on those surfaces are not 1-tough. (A graph G is said to be k-*tough* if for any separating set $S \subset V(G)$ of G, the number of components of $G - S$ is at most $\frac{1}{k}|S|$. It is easy to see that if a graph G is Hamiltonian, then G must be 1-tough, which is a well-known necessary condition for G to be Hamiltonian.)

However, for any surface F^2 with Euler characteristic $\epsilon(F^2) < 0$, we can construct a 4-connected graph on F^2 with no Hamiltonian cycle. The construction is as follows: Let Q be a *quadrangulation* on F^2, i.e., a map on F^2 with each face quadrilateral. Let $\tilde{Q}$ denote the map on F^2 obtained from Q by adding a new vertex to each face of Q and joining it to all vertices lying on the corresponding face boundary, and call it the *face subdivision* of Q. Let $S(\tilde{Q}) = V(\tilde{Q}) - V(Q)$. It is easily verified that $\tilde{Q}$ is 4-connected. However, by Euler's formula, we have $|F(Q)| = |V(Q)| - \epsilon(F^2) > |V(Q)|$. By the construction of the face subdivision, $S(\tilde{Q})$ is an independent set of $\tilde{Q}$ such that $|S(\tilde{Q})| > |V(Q)|$. So, since $\tilde{Q}$ is not 1-tough, $\tilde{Q}$ is not Hamiltonian.

How about the *torus*, an orientable surface with Euler characteristic exactly zero? By the above construction, we cannot obtain a non-1-tough graph on the torus. Actually, Nash-Williams has made the following conjecture:

Conjecture 3 (Nash-Williams [5]) *Every 4-connected torus map is Hamiltonian.*

Conjecture 3 is still open but Thomas, Yu and Zang [9] have proved that every 4-connected toroidal graph has a Hamiltonian path. Note that if we strengthen the connectivity of graphs in the problem, then the following are known: Every 5-connected toroidal graph is Hamiltonian [8] and every 5-connected triangulation on any surface with high representativity is Hamiltonian [11]. (The *representativity* of a graph G on a surface F^2 is the minimum number of intersecting points of G and ℓ, where ℓ is taken over all essential simple closed curves on F^2.)

In this paper, we shall prove the following theorem.

Theorem 4 *Every bipartite quadrangulation Q on the torus admits a simple closed curve visiting each vertex and each face of Q exactly once, but crossing no edge of Q.*

Theorem 4 immediately implies that the face subdivision of any bipartite quadrangulation on the torus is Hamiltonian, though such graphs have toughness exactly 1, where the *toughness* of a graph G is defined to be the maximum k such that G is k-tough. So, since the face subdivisions of quadrangulations are 4-connected, this contributes a partial solution of Conjecture 3 for a large family of 4-connected toroidal graphs.

Moreover, if Q is a quadrangulation on the torus and its face subdivision $\tilde{Q}$ is Hamiltonian, then every Hamiltonian cycle of $\tilde{Q}$ must visit $V(Q)$ and $S(\tilde{Q})$ alternately, since $S(\tilde{Q})$ is an independent set of $\tilde{Q}$ such that $|V(Q)| = |S(\tilde{Q})|$. Hence Theorem 4 implies the following, where the *radial graph* of a map G, denoted by $R(G)$, is a bipartite quadrangulation obtained from the face subdivision $\tilde{G}$ of G by removing all edges of G.

Corollary 5 *Let Q be a bipartite quadrangulation on the torus. Then the radial graph $R(Q)$ of Q is Hamiltonian.*

Note that Corollary 5 points out the Hamiltonicity of a large family of bipartite quadrangulations on the torus, though only the 4-regular case has been dealt with before [1]. Of course, no all bipartite quadrangulations are Hamiltonian; in fact, those with unbalanced partite sets are counterexamples.

Moreover, any bipartite quadrangulation on a surface F^2 is the radial graph $R(G)$ of some map G on F^2 with each face bounded by a cycle. Such a map G is 2-connected and has representativity at least 2, and it is sometimes called a *closed* 2*-cell embedding.*

Corollary 6 *Let G be a closed 2-cell embedding on the torus. Then the map obtained from G by taking a radial graph twice, i.e., $R(R(G))$, is Hamiltonian.*

In order to prove Theorem 4, we use an orientation of quadrangulation on the torus with each vertex of outdegree 2. We introduce such an orientation of quadrangulations on the torus in Section 2, and then prove Theorem 4 in Section 3.

2 2-Orientations of toroidal quadrangulations

An *orientation* of a graph G is an assignment of a direction to each edge of G. Let $\vec{G}$ denote the graph with the orientation and distinguish it from the undirected graph G. For a vertex v of $\vec{G}$, the *outdegree* of v is the number of directed edges outgoing from v and denoted by $\mathrm{od}(v)$, and the *indegree* of v is that of incoming edges to v and denoted by $\mathrm{id}(v)$. We say that $\vec{G}$ is a *k-orientation* or *k-oriented* if each vertex of $\vec{G}$ has outdegree exactly k. For an oriented edge $e = xy$, x is called the *origin* of e and y the *terminus.*

The following is the main result in this section.

Lemma 7 *Every quadrangulation Q on the torus admits a 2-orientation.*

In order to prove Lemma 7, we use the following proposition which is well-known as Generalized Marriage Theorem:

Proposition 8 (Corollary 11 in Section III [2]) *Let $G = (X, Y)$ be a bipartite graph with partite sets X and Y. Let $f : X \to \{0, 1, 2, \ldots\}$ be a function. Then G has a spanning subgraph H such that for any $x \in X$, $\deg_H(x) = f(x)$ and for any $y \in Y$, $\deg_H(y) = 1$ if and only if*

$$\sum_{x \in X} f(x) = |Y|$$

and for any $S \subset X$,

$$\sum_{s \in S} f(s) \leq \left| \bigcup_{s \in S} N_G(s) \right|.$$

We first prove the following proposition guaranteeing a graph G to have an orientation in which each vertex has a specified outdegree.

Proposition 9 *Let G be a graph and let $f : V(G) \to \{0, 1, 2, \ldots\}$ be a function. Then G has an f-orientation $\vec{G}$, i.e., an orientation such that $\mathrm{od}_{\vec{G}}(v) = f(v)$ for any $v \in V(G)$, if and only if*

$$\sum_{v \in V(G)} f(v) = |E(G)|,$$

and for any $S \subset V(G)$,

$$\sum_{s \in S} f(s) \geq |E([S])|,$$

where $[S]$ denotes the subgraph of G induced by S.

Proof. Suppose G has an f-orientation $\vec{G}$. Then we clearly have $\sum_{v \in V(G)} f(v) = |E(G)|$. Fix any $S \subset V(G)$. Then for any $s \in S$, $\mathrm{od}_{[\vec{S}]}(s) \leq \mathrm{od}_{\vec{G}}(s) = f(s)$, where $[\vec{S}]$ denotes the subgraph of $\vec{G}$ induced by S. Moreover, since $\sum_{s \in S} \mathrm{od}_{[\vec{S}]}(s) = |E([S])|$, we have $\sum_{s \in S} f(s) \geq |E([S])|$. Thus the sufficiency holds.

Suppose that $\sum_{v \in V(G)} f(v) = |E(G)|$, and $\sum_{s \in S} f(s) \geq |E([S])|$ for any $S \subset V(G)$. In the latter, if we let $T = V(G) - S$, then we have

$$\sum_{t \in T} f(t) \leq |E([T])| + e(S, T) \tag{1}$$

since $\sum_{v \in V(G)} f(v) = |E(G)|$, where $e(S, T)$ denotes the number of edges of G joining a vertex in S and a vertex in T. Define a bipartite graph $B = (V, E)$ with partite sets V and E, where $V = V(G)$ and $E = E(G)$, and $v \in V$ and $e \in E$ are adjacent in B if and only if v is an endpoint of e in G. Then the inequality (1) implies

$$\sum_{t \in T} f(t) \leq |N_B(T)|$$

for any $T \subset V$. Hence, by Lemma 8, B has a spanning subgraph F such that $\deg_F(v) = f(v)$ for each $v \in V$ and $\deg_F(e) = 1$ for each $e \in E$. Regarding each edge ve of F as a directed edge e of G outgoing from v, we can construct an f-orientation of G. ■

We prove Lemma 7 using Proposition 9.

Proof of Lemma 7. By Euler's formula, we have $|E(Q)| = 2|V(Q)|$. Then, in order to find a 2-orientation of Q by Proposition 9, we prove that for any $S \subset V(Q)$, $|E([S])| \leq 2|S|$. Suppose that $[S]$ consists of k components $S_1, \ldots, S_k$. Then each S_i induces a map on the torus with each face bounded by an even number of edges, and hence $|E(S_i)| \leq 2|S_i|$ for each i, by Euler's formula. Therefore we have

$$|E([S])| = \sum_{i=1}^{k} |E(S_i)| \leq 2 \sum_{i=1}^{k} |S_i| = 2|S|.$$

Hence we are done. ■

Similarly we can prove that every *triangulation* on the torus has a 3-orientation. Moreover, triangulations on surfaces with non-negative Euler characteristics have an orientation with outdegree 3 except a few vertices; see [3, 4] for some related results.

3 Finding a vertex-face curve

Let Q be a bipartite quadrangulation on the torus. Let $\mathcal{L}_m = \{l_1, \dots, l_m\}$ be a set of disjoint simple closed curves on the torus. We say that $\mathcal{L}_m$ is a *vertex-face m-family* for Q if each vertex of Q is visited exactly once by a member of $\mathcal{L}_m$ and each face of Q is visited exactly once by a member of $\mathcal{L}_m$ but every l_i crosses no edge of Q transversely. In particular, when $m = 1$, the unique element of $\mathcal{L}_1$ is called a *vertex-face curve* for Q.

Let Q be a bipartite quadrangulation on the torus with a bipartition $V(Q) = B \cup W$, where B and W are referred as *black* and *white* vertices. By Lemma 7, Q has a 2-orientation $\vec{Q}$.

Since the torus is orientable, we can give a clockwise orientation at each point on the torus simultaneously. Suppose that $e_1, \dots, e_m$ are the edges of $\vec{Q}$ incident to a vertex v, where $\deg_Q(v) = m$ and $e_1, \dots, e_m$ appear around v in this clockwise order. Since v has exactly two outgoing edges, say e_1 and e_k, we can decompose the incoming edges incident to v into two sets $\{e_2, \dots, e_{k-1}\}$ and $\{e_{k+1}, \dots, e_m\}$, where each of the two sets is called an *incoming edge class* of v, which might be an empty set.

Take any vertex $b_1 \in B$ and choose an incoming edge class E_{b_1} of b_1. Color all edges in E_{b_1} by the same color, say *red*, and choose the outgoing edge, say b_1w_1, in the right-hand side with respect to the direction of E_{b_1}. Now color by red the incoming edge class of w_1 containing b_1w_1, and choose the outgoing edge, say w_1b_2, in the left-hand side with respect to the direction of b_1w_1. We continue those procedures, choosing the right-hand outgoing edge at a black vertex in B and the left-hand outgoing edge at a white vertex in W alternately, as shown in Figure 1. We call them a *zigzag extension.* Note that we can choose an outgoing edge at each vertex unless the edge we proceed is already colored red. Hence, by a zigzag extension, we can find a directed closed walk colored red, which is called a *zigzag closed walk.* However, every red edge is not necessarily contained in the closed walk.

Moreover, if an edge e outgoing from a vertex v is colored red, then we color all edges belonging to one of the two incoming edge classes red by the same manner as the above procedures. That is, we also extend the coloring along the reverse direction of e unless the incoming edge class we have to color red is empty. The subgraph of $\vec{Q}$ induced by the red edges is called a *zigzag subgraph.* Note that if we let R be a zigzag subgraph of $\vec{Q}$, then starting any edge e of R, we get R as a zigzag subgraph of Q containing e.

When we apply a zigzag extension to $\vec{Q}$, we can take a segment l_v through a vertex v which locally separates all edges incident to v into two sets so that a single incoming edge class of v and a single outgoing edge colored red are located in one of the two sides separated by l_v, and all others are in the other side of l_v. In particular, we note that if

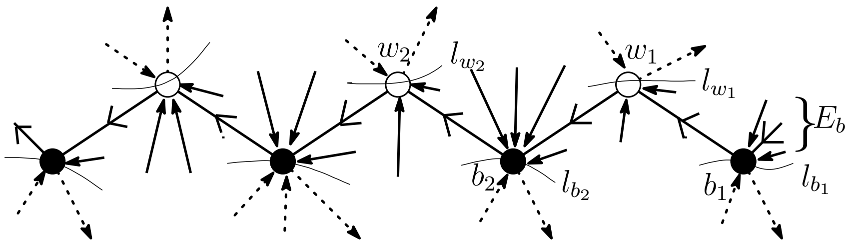

Figure 1: A zigzag extension and partitioning segments

v is a black (resp., white) vertex, then an incoming edge class and an outgoing edge are in the same side so that the direction we proceed turns right (resp., left), as shown in Figure 1. We call l_v a *partitioning segment* at v.

The following is the most important claim in this section.

Lemma 10 *Let Q be a bipartite quadrangulation on the torus. Then Q admits a vertex-face m-family $\mathcal{L}_m = \{l_1, \dots, l_m\}$ for some integer $m \geq 1$ such that*

(i) each $l_i \in \mathcal{L}_m$ is essential,

(ii) $\mathcal{L}_m$ cuts Q into m annular maps, and

(iii) each of the m annular map has an essential cycle each of whose edge joins two vertices lying on distinct boundary components.

Proof. We first prove that Q admits a vertex-face m-family for some $m \geq 1$. By Lemma 7, Q has a 2-orientation $\vec{Q}$. For each $v \in V(\vec{Q})$, take a partitioning segment l_v. Then we consider whether we can glue all l_v's for all $v \in V(\vec{Q})$ to get a required vertex-face m-family for some $m \geq 1$. Let us consider a quadrilateral face f of $\vec{Q}$ bounded by a 4-cycle $bwb'w'$, where $b, b' \in B$ and $w, w' \in W$. Observe that the 4-cycle $bwb'w'$ has seven distinct orientations shown in Figure 2, up to symmetry.

Let us consider the first example with directed edges $bw, b'w, w'b, w'b'$. Which faces does each of $l_b, l_{b'}, l_w, l_{w'}$ pass through?

(1) l_b does not separate bw and $w'b$, since a zigzag extension along $w'b$ must right-turn at b, since $b \in B$.

(2) l_w does not separate bw and $b'w$ since bw and $b'w$ belong to the same incoming edge class of w.

(3) $l_{b'}$ separates $w'b'$ and $b'w$, since a zigzag extension along $w'b'$ must right-turn at b', since $b' \in B$.

(4) $l_{w'}$ separates $w'b$ and $w'b'$ since they are two outgoing edges from w'.

Consequently, exactly two partitioning segments $l_{b'}$ and $l_{w'}$ intersect f, and hence we can glue them at the center of f.

In all other cases, we can find that exactly two partitioning segments intersect at each face, and glue them at a center of the face. Hence $\bigcup_{v \in V(\vec{Q})} l_v$ form a set of several simple closed curves visiting each vertex of $\vec{Q}$ exactly once, and each face of $\vec{Q}$ exactly once, but crossing no edge of $\vec{Q}$ transversely. Therefore, $\bigcup_{v \in V(\vec{Q})} l_v$ can be regarded as a vertex-face m-family for some $m \geq 1$.

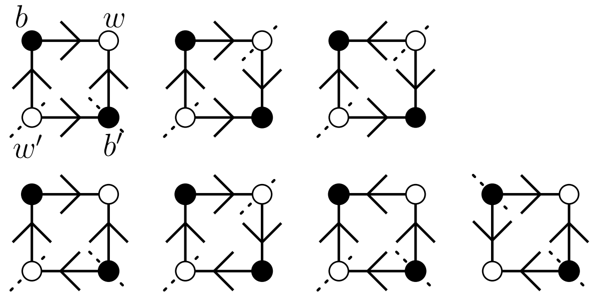

Figure 2: Possible orientations of $bwb'w'$ and partitioning segments

Now we prove (i). For contradictions, we suppose that some $l_k \in \mathcal{L}_m$ bounds a 2-cell D on the torus. We may suppose that D is *innermost*, that is, D contains no element of $\mathcal{L}_m$ other than l_k, since $\mathcal{L}_m$ is disjoint on the torus. Let $e = xy \in E(\vec{Q})$ be any edge contained in D, and suppose that e is contained in a zigzag closed walk. Observe that both x and y lie on l_k, since each vertex of $\vec{Q}$ is passed through by a member of $\mathcal{L}_m$, and l_k is innermost. However, the zigzag closed walk W containing e cannot be taken in D, since W turns left and right alternately and W cannot cross l_k transversely, a contradiction.

Since $\mathcal{L}_m = \{l_1, \ldots, l_m\}$ is disjoint on the torus and each l_i is essential, $\mathcal{L}_m$ cuts the torus into m annuli. Hence this immediately proves (ii).

Suppose that $l_1, \ldots, l_m$ appear on the torus in this cyclic order. Let $\vec{Q}_i$ be the annular map of $\vec{Q}$ bounded by l_i and l_{i+1}, where we let $l_i = l_{i+1}$ if $m = 1$. Since the annulus where $\vec{Q}_i$ embeds contains no l_j other than l_i and l_{i+1}, an endpoint of each edge of $\vec{Q}_i$ lies on either l_i or l_{i+1}. Let $W = v_1 \cdots v_p$ be a zigzag closed walk in $\vec{Q}_i$, where we put $e_j = v_j v_{j+1}$ for each j. If v_j and v_{j+1} lie on the same boundary l_i, then e_j and a segment, say l'_i, of l_i bounds a 2-cell D on the torus. By a similar argument as in (i), we can conclude that e_j can not be contained in W. Moreover, all vertices of $\vec{Q}_i$ on l_i and l_{i+1} are distinct, and W is homotopic to l_i, and hence W is a directed essential cycle. This proves (iii). ■

Lemma 10 claims that a 2-orientation of a bipartite quadrangulation Q on the torus uniquely fixes a vertex-face m-family for some $m \geq 1$, and that this uniquely partitions a bipartite quadrangulation Q into m zigzag subgraphs. However, we note that the partition depends on the 2-orientation, that is, if we take another 2-orientation of Q, then the partition of Q will change.

Lemma 11 *If a bipartite quadrangulation Q on the torus admits a vertex-face m-family for some $m \geq 2$, then G also admits a vertex-face $(m-1)$-family.*

Proof. By Lemma 10(ii), each region of the torus cut along $l_1, \ldots, l_m$ is an annulus, and hence we let A be a region bounded by distinct simple closed curves $l_1, l_2 \in \mathcal{L}_m$. Let $\vec{Q}_A$ be the subgraph of $\vec{Q}$ contained in A. Then, by Lemma 10(iii), $\vec{Q}_A$ has a directed essential cycle $C = v_1 \ldots v_{2h}$, where we suppose that v_{2i} lies on l_1 and v_{2i-1} on l_2 for $i = 1, \ldots, h$.

We focus on each segment of l_1 between v_{2i} and v_{2i+2}. Then we can find several vertices (possibly empty) lying on the segment. By the construction of a zigzag extension, we can conclude that those vertices are toward v_{2i+1} and v_{2i+2}, respectively, as shown in the left-hand side of Figure 3.

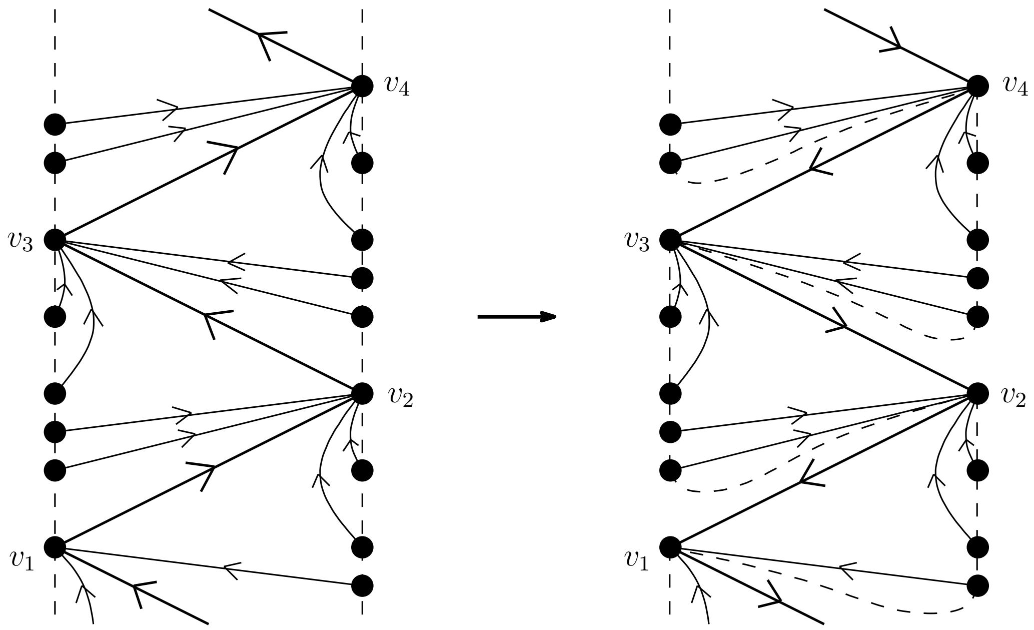

Figure 3: Reversal of C

Now reverse the direction of C in $\vec{Q}$ to get another 2-orientation. This gives no effect on all partitioning segments of vertices not on C, and makes l_1 and l_2 merge to a single essential simple closed curve $l_{1,2}$, as shown in the right-hand side of Figure 3. The resulting 2-orientation of Q gives a vertex-face $(m-1)$-family $\mathcal{L}_{m-1} = \{l_{1,2}, l_2, \ldots, l_m\}$. ■

4 Proof of the theorem

Proof of Theorem 4. Let Q be a bipartite quadrangulation on the torus. Then, by Lemma 10, Q has a vertex-face m-family for some integer $m \geq 1$. If $m = 1$, then we are done. On the other hand, if $m \geq 2$, then Q also has a vertex-face $(m-1)$-family, by Lemma 11. Applying Lemma 11 $m-1$ times, we can get a vertex-face curve for G. ■

Finally we would like to put the following.

Conjecture 12 *Let Q be any quadrangulation on the torus. Then the radial graph of Q is Hamiltonian.*

Note that in the case when Q is non-bipartite, our proof does not work, since we cannot define a zigzag extension in Q.

Observe that Conjecture 12 guarantees the Hamiltonicity of the face subdivision of any toroidal quadrangulation. Since they are 4-connected, the conjecture is weaker than Nash-Williams', and hence if Conjecture 12 is false, then so is Nash-Williams'.

References

[1] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* **4** (1973), 201–217.

[2] B. Bollobás, "*Modern Graph Theory*", Springer-Verlag, 1998.

[3] A. Nakamoto, K. Ota and T. Tanuma, Three-cycle reversions in oriented planar triangulations, *Yokohama Math. J.* **44** (1997), 123–139.

[4] A. Nakamoto and M. Watanabe, Cycle reversals in orientated plane quadrangulations and orthogonal plane partitions, *J. Geometry* **68** (2000), 200– 208.

[5] C. St. J. Nash-Williams, Unexplored and semi-explored territories in graph theory, New Directions in Graph Theory, Frank Harary, Academic Press, New York, (1973).

[6] W.T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956), 99–116.

[7] R. Thomas and X. Yu, Every 4-connected projective planar graphs are Hamiltonian, *J. Combin. Theory Ser. B* **62** (1994), 114–132.

[8] R. Thomas and X. Yu, Five-connected toroidal graphs are Hamiltonian, *J. Combin. Theory Ser. B* **69** (1997), 79–96.

[9] R. Thomas, X. Yu and W. Zang, Hamilton paths in toroidal graphs, *J. Combin. Theory Ser. B* **94** (2005), 214–236.

[10] C. Thomassen, A theorem on paths in planar graphs, *J. Graph Theory* **7** (1983), 169–176.

[11] X. Yu, Disjoint paths, planarizing cycles, and spanning walks, *Trans. Amer. Math. Soc.* **349** (1997), 1333–1358.