Spanning trees with a bounded number of branch vertices in a claw-free graph

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Abstract

Let k be a non-negative integer. A branch vertex of a tree is a vertex of degree at least three. We show two sufficient conditions for a connected claw-free graph to have a spanning tree with a bounded number of branch vertices: (i) A connected claw-free graph has a spanning tree with at most k branch vertices if its independence number is at most $2k + 2$. (ii) A connected claw-free graph of order n has a spanning tree with at most one branch vertex if the degree sum of any five independent vertices is at least $n - 2$. These conditions are best possible. A related conjecture also is proposed.

1 Introduction

In a tree, a vertex of degree one and a vertex of degree at least three is called a *leaf* and a *branch vertex*, respectively. A hamilton path can be regarded as a spanning tree with maximum degree two, a spanning tree with exactly two leaves, or a spanning tree with no branch vertex. Therefore, as one of generalized problems of a hamilton path problem, it is natural to look for conditions which ensure the existence of a spanning tree with bounded maximum degree, few leaves or few branch vertices. Many researchers have investigated the independence number conditions and the degree sum conditions for the existence of such spanning trees; bounded maximum degree [3, 9, 10, 12], few leaves [1, 11, 13] and few branch vertices [2, 4, 5, 6]. This paper mainly concerns two sufficient conditions for a claw-free graph to have a spanning tree with a bounded number of branch vertices. Several results on the hamilton path problem are known even if we restrict ourselves to claw-free graphs. For example, Matthews and Sumner [8] showed that a claw-free graph G of order n has a hamilton path if the degree sum of any three independent vertices is at least $n-2$. Moreover, Matthews and Sumner's conjecture that states a 4-connected claw-free graph is hamiltonian has motivated various problems on hamilton cycles and paths in claw-free graphs.

We first give an upper bound of an independence number $\alpha(G)$ which implies the existence of a spanning tree with a bounded number of branch vertices.

Theorem 1 *Let k be a non-negative integer. A connected claw-free graph G has a spanning tree with at most k branch vertices if $\alpha(G) \leq 2k+2$.*

This is best possible in the sense we cannot replace the upper bound of $\alpha(G)$ by $2k+3$, as shown in the next section. Next, we consider a degree sum condition for the existence of a desired spanning tree. For a graph G, let $\sigma_k(G)$ be the minimum degree sum of k independent vertices of G if $\alpha(G) \geq k$; otherwise $\sigma_k(G) = +\infty$. By Theorem 1, the degree sum condition is effective when $\alpha(G) \geq 2k+3$. We make the following conjecture.

Conjecture 2 *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{2k+3}(G) \geq n-2$, then G has a spanning tree with at most k branch vertices.*

The degree sum condition of Conjecture 2 is best possible if it is true, as shown in the next section. For $k = 0$, this conjecture corresponds to the theorem due to Matthews and Sumner [8]. Our second result shows that Conjecture 2 is true for $k = 1$.

Theorem 3 *Suppose that a connected claw-free graph G of order n satisfies $\sigma_5(G) \geq n - 2$. Then G has a spanning tree with at most one branch vertex.*

A similar degree condition for a spanning tree with few branch vertices was obtained by Gargano, Hammar, Hell, Stacho and Vaccaro [5].

Theorem 4 (Gargano et al. [5]) *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{k+3}(G) \geq n - k - 2$, then G has a spanning tree with at most k branch vertices.*

This degree condition seems slightly strong for the existence of a spanning tree with a few branch vertices. Indeed, under the same degree condition of Theorem 4, Kano, Kyaw, Matsuda, Ozeki, Saito and Yamashita [7] showed the existence of a spanning tree with at most $k+2$ leaves. Note that if a tree has at most $k+2$ leaves, then the number of branch vertices is at most k.

Theorem 5 (Kano et al. [7]) *Let k be a non-negative integer and let G be a connected claw-free graph. If $\sigma_{k+3}(G) \geq n - k - 2$, then G has a spanning tree with at most $k + 2$ leaves.*

We give proofs of Theorem 1 and Theorem 3 in Section 3 and Section 4, respectively. We prepare notation for them. For a vertex $x \in V(G)$, we denote the degree of x in G by $\deg_G(x)$ and the set of vertices adjacent to x in G by $N_G(x)$; thus $\deg_G(x) = |N_G(x)|$. For a subset $S \subset V(G)$, let $N_G(S) = \bigcup_{x \in S} N_G(x)$, and let $G - S$ denote the subgraph induced by $V(G) \setminus S$. For a tree T, let

$$\begin{aligned} L(T) &:= \{v \in V(T) : v \text{ is a leaf in } T\}, \\ B(T) &:= \{v \in V(T) : v \text{ is a branch vertex in } T\}, \\ B_i(T) &:= \{v \in B(T) : \deg_T(v) = i\}, \text{ and} \\ B_{\geq i}(T) &:= \{v \in B(T) : \deg_T(v) \geq i\}. \end{aligned}$$

Let T be a rooted tree with root r. For $u, v \in V(T)$, the unique path in T connecting u and v is denoted by uTv. For $u \in V(T) - \{r\}$, the unique neighbor of u on rTu is denoted by u^-; the *children* of u are the vertices in $N_T(u) - \{u^-\}$. We define the *distance between r and u in T*, denoted by $\text{dist}_T(r, u)$, as the length of rTu.

2 Sharpness of Theorem 1 and Conjecture 2

In Conjecture 2, the condition "$\sigma_{2k+3}(G) \geq n - 2$" is best possible if it is true. This is shown in the following example:

For each $i = 0, 1, \ldots, k$, let T_i be a triangle with $V(T_i) = \{x_i, y_i, z_i\}$. The graph G is obtained from $T_0 \cup T_1 \cup \cdots \cup T_k$ by adding k edges $z_i x_{i+1}$ for $0 \leq i \leq k-1$ and by joining each vertex of x_0, y_0, y_1, …, y_k, z_k and $k+3$ complete graphs K_{m-1}, respectively. (See Figure 1). We can take $2k+3$ independent vertices of G which consist of the k vertices $x_1, x_2 \ldots, x_k$ and $k+3$ vertices from each complete graph K_m. Then $|V(G)| = (k+3)m + 2k$ and the degree sum of these $2k+3$ vertices is $|V(G)| - 3$. Since at least one vertex in each triangle T_i with $0 \leq i \leq k$ has to be a branch vertex of any spanning tree, G has no spanning tree with at most k branch vertices.

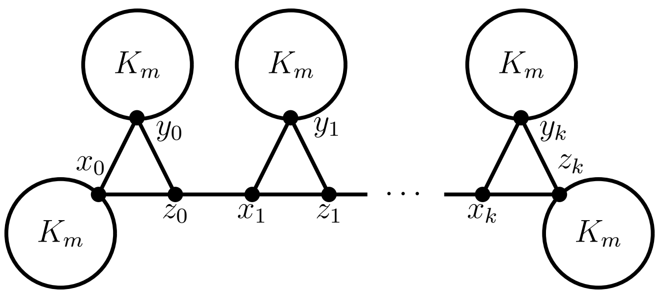

Figure 1: The graph G.

Note that the graph G also shows the sharpness of the independence number condition of Theorem 1.

3 Proof of Theorem 1

Proof of Theorem 1. Let G be a graph satisfying the assumption of Theorem 1 and having no spanning tree T with $|B(T)| \leq k$. We choose a spanning tree T of G so that

(T1) $|B(T)|$ is as small as possible,

(T2) $|L(T)|$ is as small as possible; subject to (T1), and

(T3) $|B_3(T)|$ is as small as possible; subject to (T2).

By the choice of (T2), we have the following fact.

Fact 1 *$L(T)$ is an independent set in G.*

Since G has no spanning tree T with $|B(T)| \leq k$, we obtain $|B(T)| \geq k+1$. If $B_3(T) = \emptyset$, then $\deg_T(v) \geq 4$ for every $v \in B(T)$ and hence

$$|L(T)| = 2 + \sum_{v \in B(T)} \big(\deg_T(v) - 2\big) \geq 2 + 2|B(T)| \geq 2k + 4.$$

This together with Fact 1 contradicts $\alpha(G) \leq 2k+2$. Hence there exists a vertex r in $B_3(T)$. Consider a spanning tree T as a rooted tree with root r. Let $B_3^* := B_3(T) - \{r\} := \{v_1, \ldots, v_m\}$, and let $b_i := \mathrm{dist}_T(r, v_i)$ for $i = 1, \ldots, m$. We may assume that $b_1 \leq \cdots \leq b_m$. We call a sequence $(b_1, \ldots, b_m)$ *a distance sequence of* (T, r). Note that spanning trees satisfying (T1)–(T3) have the same number of vertices of degree three. Choose T and r such that

(T4) the distance sequence $(b_1, b_2, \ldots, b_m)$ of (T, r) is as small as possible in lexicographic order; subject to (T1)–(T3).

Claim 2 *For every $v \in B_3^*$, two children of v are adjacent in G.*

Proof. Suppose that there exists $v \in B_3^*$ whose two children are independent. Since G is claw-free, at least one of two children of v, say w, is adjacent to v^-. Then $T' := T - \{vw\} + \{wv^-\}$ is a spanning tree of G, which contradicts the choice of T. In fact, if $\deg_T(v^-) \geq 3$, then T' contradicts (T1) as $B(T') = B(T) - \{v\}$; if $\deg_T(v^-) = 2$, then T' contradicts (T4) as $\mathrm{dist}_{T'}(r, v^-) < \mathrm{dist}_T(r, v)$. □

Define $X = L(T) \cup B_3^*$.

Claim 3 *For any $u_1, u_2 \in X$ with $u_1u_2 \in E(G)$, the following three statements hold.*

(i) *$u_1 \notin V(rTu_2)$ and $u_2 \notin V(rTu_1)$.*

(ii) *For any $w \in V(u_1Tu_2) - \{u_1, u_2\}$, $\deg_T(w) = 2$ or $\deg_T(w) \geq 4$.*

(iii) *For any* $w \in V(u_1Tu_2) \cap B_{\geq 4}(T)$, *let* $\{w_i\} = N_T(w) \cap V(u_iTw)$ *for* $i = 1, 2$. *Then* $N_G(w_i) \cap N_T(w) = \{w_{3-i}\}$ *for* $i = 1, 2$ *and* $G[N_T(w) - \{w_1, w_2\}]$ *is complete.*

Proof. (i) By Fact 1 and symmetry, we may consider only the case when $u_1 \in B_3^*$ and $u_1 \in V(rTu_2)$. By Claim 2, two children x_1, x_2 of u_1 are adjacent. Then $T' := T - \{u_1x_1, u_1x_2\} + \{u_1u_2, x_1x_2\}$ satisfies $|B(T')| < |B(T)|$, which contradicts (T1). Thus (i) holds.

(ii) Suppose that there exists $w \in V(u_1Tu_2) - \{u_1, u_2\}$ with $\deg_T(w) = 3$. Let x be a neighbor of w in $V(u_1Tu_2)$. Then $T - \{wx\} + \{u_1u_2\}$ contradicts (T1) and hence (ii) is shown.

(iii) Suppose that there exists $w' \in (N_G(w_i) \cap N_T(w)) - \{w_{3-i}\}$ for some $i = 1, 2$. Then $T' := T - \{ww_i, ww'\} + \{w_iw', u_1u_2\}$ has fewer branch vertices if $\deg_T(w) = 4$, or fewer leaves if $u_1 \in L(T)$ or $u_2 \in L(T)$, or fewer vertices with degree three in T' than T; otherwise. This is a contradiction. Hence $(N_G(w_i) \cap N_T(w)) - \{w_{3-i}\} = \emptyset$. Since G is claw-free, $w_1w_2 \in E(G)$, which implies $N_G(w_i) \cap N_T(w) = \{w_{3-i}\}$ for each $i = 1, 2$. If there exist two nonadjacent vertices $x_1, x_2 \in N_T(w) - \{w_1, w_2\}$, then for each $i = 1, 2$, $\{w, w_i, x_1, x_2\}$ is an induced claw. This contradicts the assumption of this theorem. Therefore $G[N_T(w) - \{w_1, w_2\}]$ is complete. □

For $u, v \in X$ such that $uv \in E(G)$, let $f(u, v)$ be a unique vertex in $V(uTv) \cap V(rTu) \cap V(rTv)$. By Claim 3 (i) and (ii), $f(u, v) \neq u, v, r$ and $f(u, v) \in B_{\geq 4}(T)$.

Claim 4 $G[X]$ *has no* P_3 *as a subgraph.*

Proof. Suppose that there exists a path $u_1u_2u_3$ in $G[X]$. Let w be the unique vertex in $V(u_1Tu_2) \cap V(u_2Tu_3) \cap V(u_3Tu_1)$. Since $u_i \in X$ for each $i = 1, 2, 3$, it follows from Claim 3 (ii) that $u_3 \notin V(u_1Tu_2)$ and $u_1 \notin V(u_2Tu_3)$. Moreover, by Claim 3 (i), we have $u_2 \notin V(rTu_1) \cup V(rTu_3)$. Thus $u_h \notin V(u_iTu_j)$ for each $\{h, i, j\} = \{1, 2, 3\}$. Consequently, $w \neq u_i$ for $i = 1, 2, 3$, and hence there exist $w_i \in N_T(w) \cap V(wTu_i)$ for $i = 1, 2, 3$. Note that $w_i \neq w_j$ for any $1 \leq i < j \leq 3$. Applying Claim 3 (iii) to u_1 and u_2, we have $w_1w_2 \in E(G)$ and $w_2w_3 \notin E(G)$. On the other hand, by applying Claim 3 (iii) to u_2 and u_3, we obtain $w_2w_3 \in E(G)$, a contradiction. □

Claim 5 *Let* u_1, u_2, u_3, u_4 *be four vertices in* X. *If* $u_1u_2 \in E(G)$ *and* $u_3u_4 \in E(G)$, *then* $f(u_1, u_2) \neq f(u_3, u_4)$.

Proof. Suppose that $w := f(u_1, u_2) = f(u_3, u_4)$. Note that $w \neq r$ and $w \neq u_i$ for each $1 \leq i \leq 4$ since $\deg_T(w) \geq 4$ (by Claim 3 (ii)), $\deg_T(r) = 3$ and $\deg_T(u_i) \in \{1, 3\}$. Let $\{w_i\} = N_T(w) \cap V(wTu_i)$ for each $1 \leq i \leq 4$. Since $w \in V(rTu_i)$ for $1 \leq i \leq 4$ and $w \neq r$, there exists w^- such that $w^- \neq w_i$. Applying Claim 3 (iii) to u_1 and u_2, we obtain $N_G(w_i) \cap N_T(w) = \{w_{3-i}\}$ for each $i = 1, 2$. Moreover, applying Claim 3 (iii) to u_3 and u_4, we obtain $N_G(w_i) \cap N_T(w) = \{w_{7-i}\}$ for $i = 3, 4$. Hence $w^- \notin N_G(w_1) \cup N_G(w_3)$. Thus, $\{w, w^-, w_1, w_3\}$ is an induced claw, a contradiction. □

Let Y be a maximum independent subset of X. By the maximality of Y and by Claim 4, every $x \in X - Y$ is adjacent to a unique vertex $y \in Y$ in G. Since $f(x, y) \in V(xTy)$, it follows from Claim 3 (ii) that $\deg_T(f(x, y)) \geq 4$. Therefore by Claim 5, we can define an injective mapping g from $X - Y$ into $B_{\geq 4}(T)$ by $g(x) := f(x, y)$. Hence $|X - Y| = |g(X - Y)| \leq |B_{\geq 4}(T)|$.

Let $Z := Y \cup \big(g(X - Y) \cap B_4(T)\big)$. To complete the proof, we prove that Z is an independent set of order at least $2k + 3$.

Claim 6 $|Z| \geq 2k + 3$.

Proof. Since T is a tree, it follows that

$$|L(T)| = 2 + \sum_{v \in B(T)} \big(\deg_T(v) - 2\big) \geq 2 + |B_3(T)| + 2|B_4(T)| + 3|B_{\geq 5}(T)|.$$

Hence we obtain

$$\begin{aligned}
|Z| &= |Y| + |g(X - Y) \cap B_4(T)| \geq |Y| + |X - Y| - |B_{\geq 5}(T)| \\
&= |L(T)| + |B_3(T)| - 1 - |B_{\geq 5}(T)| \geq 1 + 2\big(|B_3(T)| + |B_4(T)| + |B_{\geq 5}(T)|\big) \\
&= 1 + 2|B(T)| \geq 2k + 3. \qquad \square
\end{aligned}$$

Claim 7 *No vertex of Y is adjacent to any vertex of $g(X - Y) \cap B_4(T)$ in G.*

Proof. Suppose that $u_1 w \in E(G)$ for some $u_1 \in Y$ and for some $w \in g(X - Y) \cap B_4(T)$. Let $u_2, u_3 \in X$ such that $w = f(u_2, u_3)$ and let $\{w_i\} = N_T(w) \cap V(u_i T w)$ for each $i = 2, 3$. Note that $u_2 u_3 \in E(G)$. By Claim 3 (iii), we have $w_2 w_3 \in E(G)$.

We first show that $u_2, u_3 \notin V(rTu_1)$. To the contrary, assume that $u_2 \in V(rTu_1)$. Then $T' := T - \{ww_2, ww_3\} + \{wu_1, w_2 w_3\}$ is a spanning tree with $|B(T)| = |B(T')|$ and $\deg_{T'}(w) = 3$. If $u_1 \in L(T)$, then $L(T') = L(T) - \{u_1\}$, which contradicts (T2).

Therefore $u_1 \in B_3^*$ and hence $|L(T)| = |L(T')|$, $|B_3(T)| = |B_3(T')|$ and $\deg_{T'}(u_1) = 4$. Since $u_1 \in B_3^*$ and $\mathrm{dist}_T(r, w) < \mathrm{dist}_T(r, u_1)$, there exists an integer l such that $b_{l-1} \leq \mathrm{dist}_T(r, w) < b_l$. Let $(b'_1, b'_2, \ldots, b'_m)$ be the distance sequence of (T', r). By the definition of T', we have $b'_i = b_i$ for every $1 \leq i \leq l-1$ and $b'_l = \mathrm{dist}_{T'}(r, w) = \mathrm{dist}_T(r, w) < b_l$, contradicting (T4). Therefore $u_2 \notin V(rTu_1)$. Similarly we obtain $u_3 \notin V(rTu_1)$. Moreover, by Claim 3 (ii), $u_1 \notin V(u_2Tu_3) - \{u_2, u_3\}$.

Hence there exists a unique vertex $z \in V(u_1Tu_2) \cap V(u_1Tu_3) \cap V(u_2Tu_3)$, which satisfies $z \neq u_i$ for any $i = 1, 2, 3$. Let $z_i \in N_T(z) \cap V(u_iTz)$ for $i = 1, 2, 3$. By Claim 3 (ii), we have $z \in B_{\geq 4}(T)$, and so it follows from Claim 3 (iii) and the fact $u_2u_3 \in E(G)$ that $z_2z_3 \in E(G)$

Suppose that $\deg_T(z) = 4$ or 5. For $z_4 \in N_T(z) - \{z_1, z_2, z_3\}$, by Claim 3 (iii), we have $z_1z_4 \in E(G)$. Then $T' = T - \{zz_1, zz_3, zz_4\} + \{u_1w, z_1z_4, u_2u_3\}$ is a spanning tree with $B(T') = B(T) - \{z\}$, a contradiction.

Suppose that $\deg_T(z) \geq 6$. Then $T' = T - \{zz_2, zz_3\} + \{z_2z_3, u_1w\}$ is a spanning tree with $B(T') = B(T)$. If $u_1 \in L(T)$, then $L(T') = L(T) - \{u_1\}$, contradicting (T2). If $u_1 \in B_3^*$, then $L(T') = L(T)$ and $B_3(T') = B_3(T) - \{u_1\}$, contradicting (T3). This completes the proof of Claim 7. □

Claim 8 *$g(X - Y) \cap B_4(T)$ is an independent set.*

Proof. Suppose that $wz \in E(G)$ for some $w, z \in g(X-Y) \cap B_4(T)$. Let $u_1, u_2 \in X$ such that $w = f(u_1, u_2)$. Since $w \notin V(rTz)$ or $z \notin V(rTw)$, without loss of generality, we may assume that $w \notin V(rTz)$, implying $w^- \in V(zTw)$. Let $\{w_i\} = N_T(w) \cap V(wTu_i)$ for $i = 1, 2$ and let $\{w_3\} = N_T(w) - \{w^-, w_1, w_2\}$. By Claim 3 (iii), $w_1w_2 \in E(G)$ and $w^-w_3 \in E(G)$. Then $T' := T - \{ww_1, ww^-, ww_3\} + \{zw, w^-w_3, u_1u_2\}$ is a spanning tree with $B(T') = B(T) - \{w\}$, a contradiction. □

Since Y is independent, it follows from Claims 6–8 that Z is an independent set of order at least $2k + 3$. This contradicts the assumption, and completes the proof of Theorem 1. □

4 Proof of Theorem 3

In this section, we give a proof of Theorem 3.

Proof of Theorem 3. Let G be a graph satisfying the assumption of Theorem 3, but G has no spanning tree with at most one branch vertex. By Theorem 5 with $k = 2$, G has a spanning tree T with at most four leaves. Therefore we obtain $|B(T)| = |B_3(T)| = 2$. Let P be a path joining two branch vertices. Choose such a spanning tree T of G with property that P is as short as possible.

Let x_1, x_2, x_3, x_4 be four leaves of T and let x_5, y be two branch vertices of T such that $x_5 \notin V(x_iTy)$ for $i = 1, 2$. For $1 \leq i \leq 4$, let P_i be a path of $T - V(P)$ containing an end-vertex x_i, and let y_i be the other end-vertex of P_i. (See Figure 2.) For each path P and P_i, we give a direction from x_5 to y, and x_i to y_i, respectively.

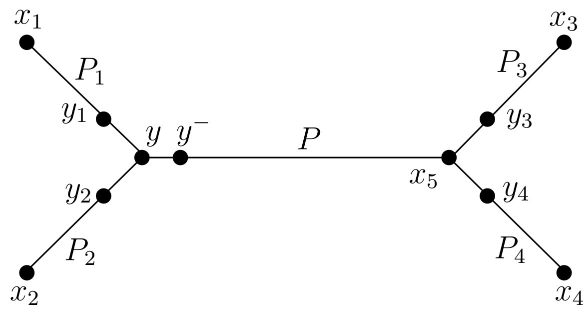

Figure 2: The spanning tree T.

By the choice of T, we obtain the following claim.

Claim 1 *$y_1y_2, y_3y_4 \in E(G)$ and $y_iy^-, y_{i+2}x_5^+ \notin E(G)$ for $i = 1, 2$.*

Proof. Suppose that $y_iy^- \in E(G)$ for some $i = 1, 2$, where $y^- \in V(P) \cap N_T(y)$. Then $T' = T - \{yy_i\} + \{y_iy^-\}$ is a spanning tree such that $B(T') = \{x_5, y^-\}$ and $|V(x_5T'y^-)| < |V(P)|$. This contradicts the choice of T. Therefore $y_iy^- \notin E(G)$ for any $i = 1, 2$. Since G is claw-free, $y_1y_2 \in E(G)$. Similarly, we obtain $y_3y_4 \in E(G)$ and $y_{i+2}x_5^+ \notin E(G)$ for any $i = 1, 2$. □

Claim 2 *No edge connects two vertices in $\{x_1, x_2, x_3, x_4, x_5, y\}$, except for x_5y.*

Proof. Suppose that $x_ix_j \in E(G)$ for some $1 \leq i < j \leq 5$. We first consider the case when $j \leq 4$. Then $T - \{yy_i\} + \{x_ix_j\}$ is a spanning tree with one branch vertex, a contradiction. Therefore $\{x_1, x_2, x_3, x_4\}$ is an independent set of G. Hence we obtain $j = 5$. Let $T' := T - \{yy_i\} + \{x_ix_5\}$ if $i = 1, 2$; otherwise $T' := T - \{x_5y_3, x_5y_4\} + \{x_ix_5, y_3y_4\}$. In either case, T' is a spanning tree with $|B(T')| = 1$, a contradiction. Therefore $x_5x_i \notin E(G)$ for $1 \le i \le 4$. Similarly, $yx_i \notin E(G)$ for $1 \le i \le 4$, and this completes the proof. $\square$

By Claim 2, $\{x_1, x_2, x_3, x_4, x_5\}$ is an independent set of G. Now we consider the degree sum of it. In particular, we shall deal with the degree sum by partitioning T into five paths, P_1, P_2, P_3, P_4 and P.

Claim 3 *$(N_G(x_h) \cap V(P_h))^- \cap N_G(x_i) = \emptyset$ for any $1 \le h \le 4$ and $1 \le i \le 5$ with $i \ne h$.*

Proof. Suppose that there exists $v \in (N_G(x_h) \cap V(P_h))^- \cap N_G(x_i)$. For $h = 1, 2$, let $T' := T - \{vv^+, yy_h\} + \{x_hv^+, x_iv\}$ and for $h = 3, 4$, let $T' := T - \{vv^+, x_5y_h\} + \{x_hv^+, x_iv\}$. Then T' is a spanning tree with $|B(T')| = 1$, a contradiction. $\square$

Claim 4 *$(N_G(x_i) \cap V(P_h)) \cap N_G(x_j) = \emptyset$ for any $1 \le h \le 4$ and $1 \le i < j \le 5$ with $i \ne h$ and $j \ne h$.*

Proof. Suppose that there exists $v \in (N_G(x_i) \cap V(P_h)) \cap N_G(x_j)$ for some $1 \le i < j \le 5$ with $i \ne h$ and $j \ne h$. Assume $j \le 4$. Without loss of generality, we may assume that $h = 1$. Then $j = 3$ or 4, and $T - \{yy_1, x_5y_j\} + \{vx_i, vx_j\}$ is a spanning tree with one branch vertex, a contradiction. Thus we may consider the case when $j = 5$. We divide this case into further two cases depending on h. Without loss of generality, we may assume that $h = 1$ or $h = 3$.

Case 1. $h = 1$.

By Claim 2, we have $v \ne x_1$ and so there exists v^-. Since $\{vv^-, vx_i, vx_5\}$ is not a claw and $x_ix_5 \notin E(G)$, we obtain $x_iv^- \in E(G)$ or $x_5v^- \in E(G)$. Let h_1, h_2 be integers such that $\{h_1, h_2\} = \{i, 5\}$ and $x_{h_1}v^- \in E(G)$. Then $T - \{vv^-, yy_1\} + \{x_{h_1}v^-, x_{h_2}v\}$ is a spanning tree with one branch vertex, a contradiction.

Case 2. $h = 3$.

Since $v \ne x_3$, there exists v^-. Since $\{vv^-, vx_i, vx_5\}$ is not a claw and $x_ix_5 \notin E(G)$, we obtain $x_iv^- \in E(G)$ or $x_5v^- \in E(G)$. Let h_1, h_2 be integers such that $\{h_1, h_2\} = \{i, 5\}$ and $x_{h_1}v^- \in E(G)$. If $h_1 = i$ or if $h_1 = 5$ and $i = 1$ or 2, then $T - \{vv^-, x_5y_3, x_5y_4\} + \{x_{h_1}v^-, x_{h_2}v, y_3y_4\}$ is a spanning tree with one branch vertex. Hence we may assume that $x_5v^- \in E(G)$ and $i = 4$.

Since $\{x_5v^-, x_5y_4, x_5x_5^+\}$ is a not claw, it follows from Claim 1 that $v^-y_4 \in E(G)$ or $v^-x_5^+ \in E(G)$. If $v^-y_4 \in E(G)$, then $T - \{vv^-, x_5y_4\} + \{v^-y_4, vx_4\}$ is a spanning tree with one branch vertex, a contradiction. Assume $v^-x_5^+ \in E(G)$. Then $T' := T - \{vv^-, x_5y_3\} + \{v^-x_5^+, vx_4\}$ is a spanning tree and $|B(T')| = 1$ if $x_5^+ = y$; otherwise $B(T') = \{y, x_5^+\}$ and $|V(x_5^+T'y)| < |V(P)|$. This contradicts the choice of T, and completes the proof. □

Claim 5 *$y_j \notin N_G(x_i)$ for any $1 \le i \le 5$ and $1 \le j \le 2$ with $i \ne j$.*

Proof. Suppose that there exists an edge x_iy_j for some $1 \le i \le 5$ and $1 \le j \le 2$ with $i \ne j$. Then $T' := T - \{yy_j\} + \{x_iy_j\}$ is a spanning tree with $|B(T')| = 1$, a contradiction. □

By Claim 3 and Claim 4, the five sets $(N_G(x_h) \cap V(P_h))^-$, $N_G(x_i) \cap V(P_h)$ $(1 \le i \le 5,\ i \ne h)$ are pairwise disjoint. Moreover, by Claim 5, y_h is not contained in these five sets if $h = 1, 2$. Therefore, for $h = 1, 2$, we obtain

$$\begin{aligned} \sum_{i=1}^{5} |N_G(x_i) \cap V(P_h)| &= |(N_G(x_h) \cap V(P_h))^-| + \sum_{i \ne h} |N_G(x_i) \cap V(P_h)| \\ &\le |V(P_h) - \{y_h\}| = |V(P_h)| - 1, \end{aligned} \tag{1}$$

and for $h = 3, 4$,

$$\sum_{i=1}^{5} |N_G(x_i) \cap V(P_h)| \le |V(P_h)|. \tag{2}$$

Suppose that there exists $v \in N_G(x_i) \cap V(P)$ for some $1 \le i \le 4$. Then by Claim 2, we have $v \ne y$, and hence $T - \{yy_i\} + \{vx_i\}$ contradicts the choice of T. Hence $N_G(x_i) \cap V(P) = \emptyset$ for $1 \le i \le 4$. Thus we deduce

$$\sum_{i=1}^{5} |N_G(x_i) \cap V(P)| = |N_G(x_5) \cap V(P)| \le |V(P)| - 1. \tag{3}$$

Summing the inequalities (1)–(3) yields $\sum_{i=1}^{5} \deg_G(x_i) \le |T| - 3 = n - 3$. This contradicts the assumption of this theorem. □

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