4-connected triangulations and 4-orderedness

Raiji Mukae

Faculty of Education and Human Sciences,
Yokohama National University
79-7 Tokiwadai, Hodogaya-ku,
Yokohama 240-8501, Japan
e-mail: mkerij@gmail.com

Kenta Ozeki

National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku,
Tokyo 101-8430, Japan
e-mail: ozeki@nii.ac.jp

Abstract

For a positive integer $k \geq 4$, a graph G is called *k-ordered*, if for any ordered set of k distinct vertices of G, G has a cycle that contains all the vertices in the designated order. Goddard [3] showed that every 4-connected triangulation of the plane is 4-ordered. In this paper, we improve this result; every 4-connected triangulation of any surface is 4-ordered. Our proof is much shorter than the proof by Goddard.

1 Introduction

A graph G is called *k-ordered* for an integer $4 \leq k \leq |V(G)|$, if for any ordered set of k distinct vertices of G, G has a cycle that contains all the vertices in the designated order. This topic has been extensively studied; see for example [1, 2, 4, 5]. Considering the topic on the concept of "k-linked", it is known that the high connectivity guarantees the k-orderedness, in particular, every $10k$-connected graph is k-ordered [9]; see also [1]. However, little is known about the minimum connectivity that implies 4-ordered. Faudree [1] proposed the following question;

If G is a 6-connected graph, is G 4-ordered?

This question is still open. However, if we restrict to a triangulation of the plane, Goddard [3] showed that smaller connectivity assumption guarantees the 4-orderedness.

Theorem 1 (Goddard [3]) *Let G be a 4-connected triangulation of the plane. Then G is 4-ordered.*

In this paper, we extend this result to other surfaces.

Theorem 2 *Let G be a 4-connected triangulation of any surface. Then G is 4-ordered.*

The proof of Theorem 2 is very different from that of Theorem 1. In [3], it is tried to find a contractible edge and to use the induction on $|V(G)|$. On the other hand, in this paper, we do not use the induction method. For given four vertices, we directly try to find a cycle containing such vertices in a given order. In fact, the proof of this paper is much shorter than the proof in [3].

2 Proof of Theorem 2

In order to prove Theorem 2, we use the following theorem, which is famous as "2-path Theorem". For a graph embedded on a surface,

Theorem 3 (Seymour [7], Shiloach [8], Thomassen [10]) *Let G be a 4-connected graph, and suppose that four vertices s_1, t_1, s_2, t_2 are given. Then, either*

(1) *there are two disjoint paths P_1, P_2 such that P_i connects s_i and t_i for $i = 1, 2$; or*

(2) *there exists an embedding of G into the plane so that one face boundary cycle contains four vertices s_1, s_2, t_1, t_2 in the clockwise order.*

Proof of Theorem 2. Let G be a 4-connected triangulation of a surface and let $\{x_1, x_2, x_3, x_4\}$ be an ordered set of four distinct vertices of G. We shall find a cycle containing x_1, x_2, x_3, x_4 in this order.

Case 1. For any $1 \leq i \leq 4$, $x_i x_{i-1} \in E(G)$ or $x_i x_{i+1} \in E(G)$.

In this case, we may assume that $x_4 x_1, x_2 x_3 \in E(G)$. Suppose that there exist no two disjoint paths P_1 and P_2 such that P_i connects x_{2i-1} and x_{2i} for $i = 1, 2$. It follows from Theorem 3 that G can be embedded into the plane so that one face boundary contains x_1, x_3, x_2, x_4 in the clockwise order. If G is a triangulation of a surface which is not plane, then this is a contradiction, because G cannot be embedded into the plane. On the other hand, if G is a triangulation of the plane, then any embedding of G into the plane cannot have a non-triangular face, which is also a contradiction. In either case, we have a contradiction. Therefore there exist two disjoint paths P_1 and P_2 such that P_i connects x_{2i-1} and x_{2i} for $i = 1, 2$. So, $P_1 \cup x_2 x_3 \cup P_2 \cup x_4 x_1$ is a cycle containing x_1, x_2, x_3, x_4 in this order. □

Case 2. For some $1 \leq i \leq 4$, $x_i x_{i-1} \notin E(G)$ and $x_i x_{i+1} \notin E(G)$.

We may assume that $i = 2$, that is, $x_1 x_2 \notin E(G)$ and $x_2 x_3 \notin E(G)$. Since G is a triangulation, we can take a cycle C through all the vertices in $N_G(x_2)$, which is called a *link of* x_2; see Theorem 3 in [6]. By the assumption of Case 2, note that $x_1, x_3 \notin V(C)$. Since G is 4-connected, we can find four pairwise internally-disjoint paths P_1, P_2, P_3, P_4 such that P_1 and P_2 connect x_1 and $V(C)$, and P_3 and P_4 connect x_3 and $V(C)$. We may assume that $|V(P_i) \cap V(C)| = 1$, say $\{y_i\} = V(P_i) \cap V(C)$. Notice that $y_i \neq y_j$ for any $1 \leq i < j \leq 4$. Since C is a link of x_2, note also that $x_2 \notin V(P_i)$ for every $1 \leq i \leq 4$. Moreover, replacing the indices of P_1 and P_2 if necessary, we may also assume that the subpath of C from y_1 to y_3, say C_1, does not intersect with the subpath of C from y_2 to y_4, say C_2. Let H be the cycle of G which consists of $P_1 \cup C_1 \cup P_3 \cup P_4 \cup C_2 \cup P_2$.

Since C is a link of x_2, $x_2 y_i \in E(G)$ for any $1 \leq i \leq 4$. Hence x_2 can be "inserted" into H instead of C_1, that is, we can find the cycle $\big(H - C_1\big) \cup y_1 x_2 \cup x_2 y_2$. By the same way, we can also insert x_2 into H instead of C_2.

If $x_4 \in V(H)$, say $x_4 \in V(P_2 \cup C_2 \cup P_4)$ by the symmetry, then the cycle obtained by inserting x_2 instead of C_1 is the desired cycle. (See the left side of Figure 1). Thus we may assume that $x_4 \notin V(H)$. Since G is 4-connected, we can find three pairwise internally disjoint paths Q_1, Q_2, Q_3 from x_4 to $V(H)$ in $G - x_2$. We may also assume that for $i = 1, 2$, $V(Q_i) \cap V(P_2 \cup C_2 \cup P_4) \neq \emptyset$, say $\{z_i\} = V(Q_i) \cap V(P_2 \cup C_2 \cup P_4)$. Let H' be the subpath of $P_2 \cup C_2 \cup P_4$ between z_1 and z_2. Then $(H - H') \cup Q_1 \cup Q_2$ is the cycle containing x_1, C_1, x_3, x_4 in this order. Hence the cycle obtained by inserting x_2 instead of C_2 is also the desired cycle. (See the right side of Figure 1). This completes the proof of Theorem 2. $\square$

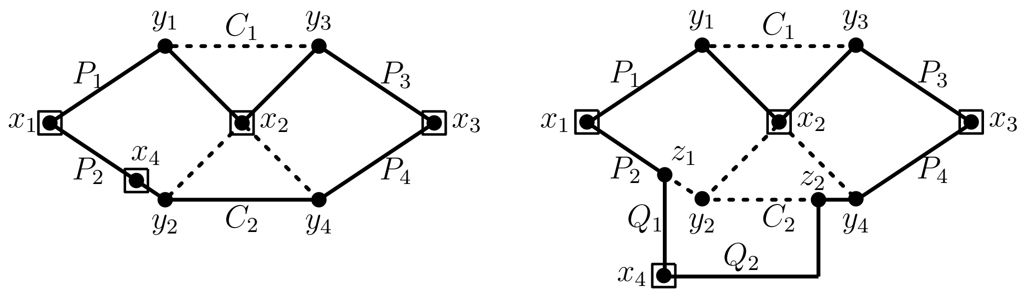

Figure 1: The desired cycles.

References

[1] R.J. Faudree, Survey of results on k-ordered graphs, Discrete Math. **229** (2001) 73–87.

[2] R.J. Faudree, R.J. Gould, A.V. Kostochka, L. Lesniak, I. Schiermeyer and A. Saito, Degree conditions for k-ordered hamiltonian graphs, J. Graph Theory **42** (2003) 199–210.

[3] W. Goddard, 4-connected maximal planar graphs are 4-ordered, Discrete Math. **257** (2002) 405–410.

[4] K. Mészáros, On 3-regular 4-ordered graphs, Discrete Math. **308** (2008) 2149–2155.

[5] L. Ng and M. Schultz, k-ordered hamiltonian graphs, J. Graph Theory **24** (1997) 45–57.

[6] G. Ringel, Non-existence of graph embeddings, in "Theory and Applications of Graphs", Lecture Notes Math. **642**, Springer-Verlag, Berlin (1978) 465–476.

[7] P.D. Seymour, Disjoint paths in graphs, Discrete Math. **29** (1980) 293–309.

[8] Y. Shiloach, A polynomial solution to the undirected two paths problem, J. ACM **27** (1980) 445–456.

[9] R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, European J. Combin. **26** (2005) 309–324.

[10] C. Thomassen, 2-linked graph, European J. Combin. **1** (1980) 371–378.