# 2- and 3-factors of graphs on surfaces

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**ABSTRACT**

It has been conjectured that any 5-connected graph embedded in a surface $\Sigma$ with sufficiently large face-width is hamiltonian. This conjecture was verified by Yu for the triangulation case, but it is still open in general. The conjecture is not true for 4-connected graphs.

In this paper, we shall study the existence of 2- and 3-factors in a graph embedded in a surface $\Sigma$. A hamiltonian cycle is a special case of a 2-factor. Thus it is quite natural to consider the existence of these factors. We give an evidence to the conjecture in a sense of the existence of a 2-factor. In fact, we only need the 4-connectivity with minimum degree at least 5. In addition, our face-width condition is not huge. Specifically, we prove the following two results. Let $G$ be a graph embedded in a surface $\Sigma$ of Euler genus $g$.

(1) If $G$ is 4-connected and minimum degree of $G$ is at least 5, and furthermore, face-width of $G$ is at least $4g - 12$, then $G$ has a 2-factor.

(2) If $G$ is 5-connected and face-width of $G$ is at least $\max\{44g - 117, 5\}$, then $G$ has a 3-factor.

The connectivity condition for both results are best possible. In addition, the face-width conditions are necessary too.

**Keywords**: Hamiltonian cycle, graphs on a surface, and factors.

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# 1 Introduction

In 1956 Tutte [19] proved that every 4-connected planar graph is hamiltonian. This is perhaps one of the most celebrated theorems of Tutte. For a short proof see [14]. Thomas and Yu [11] extended Tutte's theorem to projective planar graphs. This result does not extend to 3-connected planar graphs since there exist planar triangulations on $n$ vertices whose longest cycle is of length $O(n^{\alpha})$, where $\alpha = \log 2/\log 3 \approx 0.63$; cf. [8].

As every graph can be embedded on some surface, these results on hamiltonian cycles do not generalize to surfaces of higher genera even for 1000-connected graphs. An additional modest condition on the face-width does not help either. Archdeacon, Hartsfield, and Little [2] proved that for each $k$ there exists a $k$-connected triangulation of an orientable surface having face-width $k$ in which every spanning tree has a vertex of degree at least $k$. In particular, such graphs are far from being hamiltonian.

If the surface is fixed and the face-width is large, the situation changes. Thomassen [17] proved that large face-width of a triangulation of a fixed orientable surface implies the existence of a spanning tree of maximum degree at most 4 and that 4 cannot be replaced by 3. It was conjectured in [17] that the additional condition that the triangulation is 5-connected implies that the graph is hamiltonian, and this was verified by Yu [20]. This result was further extended by the first author [6]. It was also observed in [17] that "5-connected" cannot be replaced by "4-connected". But it has been conjectured (see [7]) that any 5-connected graph embedded in a surface with sufficiently large face-width is hamiltonian. Again, the connectivity condition is best possible, if Euler genus of a surface is more than two. Recall that a surface has Euler genus at most two if and only if it is the sphere, the projective plane, the torus, or the Klein bottle.

Concerning 4-connected graphs embedded in a torus, a well known conjecture of Grünbaum [5] and Nash-Williams [9] asserts that every 4-connected toroidal graph is hamiltonian. While this conjecture is still open, Thomas and Yu [12] proved that every 5-connected toroidal graph has a hamiltonian cycle. Very recently, Thomas, Yu and Zang [13] proved that every 4-connected toroidal graph has a hamiltonian path, but the conjecture of Grünbaum and Nash-Williams remains widely open.

A hamiltonian cycle is a special case of a 2-factor (Recall that an *i-factor* is a spanning subgraph $H$ of a given graph $G$ such that every vertex has degree exactly $i$ in $H$). Thus it is quite natural to find a 2-factor, instead of a hamiltonian cycle. In fact, Dean and Ota [3] proved that every 4-connected graph on a torus has a 2-factor. This is an evidence for the above mentioned conjecture by Grünbaum and Nash-Williams. Note that the connectivity condition for the existence of a 2-factor is also best possible.

In this paper, we shall study the existence of a 2- and 3-factor in a 5-connected graph on a surface. We give an evidence to above mentioned conjecture for hamiltonian cycles on a surface. In fact, we only need the 4-connectivity and minimum degree at least 5. In addition, our face-width condition is not huge. Specifically, we prove the following two results.

**Theorem 1** *Let $G$ be a 4-connected graph embedded in a surface $\Sigma$ of Euler genus $g$ with $\delta(G) \geq 5$ and face-width at least $4g - 12$. Then $G$ has a 2-factor.*

**Theorem 2** *Let $G$ be a 5-connected graph of even order embedded in a surface $\Sigma$ of Euler genus $g$ with face-width is at least $\max\{44g - 117, 5\}$, then $G$ has a 3-factor.*

The connectivity condition for both results are best possible. In addition, the face-width conditions are necessary too. We shall discuss these issues in section 5.

Let us point out that our proof method does not depend on the cutting method adapted first by Thomassen [15]. This method is strong enough to prove the 5-color theorem [16] and the 5-list-color theorem [4] for a graph on a fixed surface with sufficiently large face-width. Sometimes this method needs a deep theorem in the graph minor project by Robertson and Seymour [10]. But this method needs “huge” face-width. Typically face-width needs to be as much as $2^{\Omega(g)}$. On the other hand, Theorems 1 and 2 require the face-width be linear in $g$. The proof method in this paper is completely different.

This paper is organized as follows. In Section 2, we give definitions and notations that are needed in our main proofs. In Section 3, we give a proof of Theorem 1, and in Section 4, we give a proof of Theorem 2. Finally in Section 5, we conclude the all of our conditions in Theorems 1 and 2 are needed.

## 2 Definitions and Preliminaries

For notation not defined here, we refer to the book [7]. But for the sake of completeness, let us repeat some important definitions.

A *surface* $\Sigma$ is a compact connected 2-manifold without boundary. We assume familiarity with basic notions of surface topology, like genus and Euler's formula. We define the *Euler genus* of a surface $\Sigma$ as $2-\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. An *arc* in $\Sigma$ is subset homeomorphic to $[0,1]$. An *O-arc* is a subset of $\Sigma$ homeomorphic to a circle. Note that every $O$-arc should bound a disc, so it is called a *contractible* curve. A *non-contractible* curve means a curve which is not contractible.

A graph $G$ is *embedded* in a topological space $X$ if the vertices of $G$ are distinct elements of $X$ and every edge of $G$ is a simple arc connecting the two vertices in $X$ which it joins in $G$, such that its interior is disjoint from other edges and vertices. An *Embedding* of a graph $G$ in the topological space $X$ is an isomorphism of $G$ with a graph $G'$ embedded in $X$. In this case, $G'$ is said to be a representation of $G$ in $X$. If there is an embedding of $G$ into $X$, we say that $G$ *can be embedded* into $X$. Let $G$ be a graph that is embedded in $\Sigma$. To simplify notation we do not distinguish between a vertex of $G$ and the point of $\Sigma$ used in the embedding to represent the vertex, and we do not distinguish between an edge and the arc on the surface representing it. We also consider $G$ as the union of the points corresponding to its vertices and edges.

The *face-width* or the *representativity* of a graph $G$ embedded in a non-spherical surface is the smallest possible cardinality of the intersection of $G$ with a non-contractible curve on the surface. If $G$ is embedded in a surface $\Sigma$, then we say that a subgraph $H$ is *flat* if $H$ has a cycle bounding a disk in $\Sigma$ and $H$ is contained in this disk.

If $G$ is a graph and $A$ is a set of vertices of $G$, then $G[A]$ is the subgraph of $G$ induced by $A$, that is, its vertex set is $A$ and its edge set consists of all edges in $G$ joining two vertices of $A$. For a vertex $x$ and for a set $C$ of vertices, we denote the set of neighborhood of $x$ by $N_G(x)$ and let $N_G(C) := \bigcup_{x\in C} N_G(x)$ and $d_G(C) := \sum_{x\in C} d_G(x)$. For two disjoint sets $T, C$ of vertices, we denote the number of edges between $T$ and $C$ by $e(T,C)$.

## 3 Lemmas

In this section, we give some lemmas used for our proofs. In the rest of the paper, the existence of a contractible curve is important. Thus we shall give the following useful lemma first.

**Lemma 3** *Let $G$ be a graph embedded in a surface $\Sigma$ and let $S \subset V(G)$ be a cut set of $G$, i.e, $G - S$ is not connected. Let $C_1, C_2, \dots, C_p$ be components of $G - S$. If $G - C_i$ is not flat for each $1 \leq i \leq p$, then there exists a non-contractible curve $\gamma$ of $\Sigma$ such that $\gamma \cap G \subset S$.*

**Proof.** If there exist two indices $i$ and $j$ such that neither $C_i \cup S$ nor $C_j \cup S$ is flat, then it is easy to show the existence of the desired non-contractible curve by just taking a subset of $S$ that is a cut set which separates $C_i$ and $C_j$. Thus, we may assume that all but at most one $C_i \cup S$ is flat $1 \leq i \leq p$. Without loss of generality, we may assume that $C_1 \cup S$ is not flat if such a component exists.

Let $G'$ be the graph obtained from $G - C_1$ by contracting (on $\Sigma$) each component $C_2, C_3, \dots C_p$ to one vertex. Since $C_i \cup S$ is flat for each $2 \leq i \leq p$ and $G - C_1$ is not flat, $G'$ is also not. Then there exists a non-contractible curve $\gamma$ on $\Sigma$ hitting vertices of $G'$ but not hitting edges of $G'$. Considering a curve homotopic to $\gamma$, we can find a non-contractible curve hitting only vertices in $S$, which is the desiderd one. □

Next two lemmas directly follow from the Euler's formula. Hence we omit the proof. In Lemma 4, we divide the statement depending on the value of $g$, since the statement (i) does not hold for the case where $g = 0$ and $|V(G)| \leq 2$.

**Lemma 4** *Let $G$ be a bipartite graph embedded in a surface with Euler genus $g$. (i) If $g \geq 1$, then $|E(G)| \leq 2|V(G)| + 2g - 4$. (ii) If $g = 0$, then $|E(G)| \leq 2|V(G)| - 2$.*

**Lemma 5** *Let $G$ be a graph on a surface with Euler genus $g$. If $g \geq 1$, then $|E(G)| \leq 3|V(G)| + 3g - 6$.*

We will use the following lemma in our proof of Theorem 2.

**Lemma 6** *Suppose $G$ is a 5-connected graph embedded in a surface $\Sigma$ with face-width at least five. Then $G$ does not contain $K_{2,3}$ as a subgraph.*

**Proof.** Since $G$ is 5-connected and face-width of $G$ is at least five, any cycle consisting of at most four vertices bounds a face or bounds two adjacent faces of $G$. Suppose that $G$ contains $K_{2,3}$ as a subgraph and let $\{x_1, x_2\}$ and $\{y_1, y_2, y_3\}$ be two partite sets of $K_{2,3}$. Then all the cycles $x_1y_1x_2y_2$, $x_1y_1x_2y_3$ and $x_1y_2x_2y_3$ have the above property, and hence $N_G(y_1) \subset \{x_1, x_2, y_2, y_3\}$, which contradicts 5-connectedness of $G$. □

For $S, T \subset V(G)$ with $S \cap T = \emptyset$, let $h_G(S, T)$ be the number of components $C$ of $G - (S \cup T)$ with $e(C, T) \equiv 1 \pmod 2$. Moreover, we call such a component *odd*. Let $h'_G(S, T)$ be the number of components $C$ of $G - (S \cup T)$ with $e(C, T) + |C| \equiv 1 \pmod 2$. Tutte [18] gave a necessary and sufficient condition for the existence of a factor in which the degrees of all vertices are constant. The following theorems are Tutte's results on a 2-factor and on a 3-factor, respectively. For a 2-factor, Aldred, Egawa, Fujisawa, Ota and Saito strengthened the Tutte's result. (If we take sets of vertices $S$ and $T$ satisfying Lemma 9 (i) so that $|S \cup T|$ is as small as possible, then $S$ and $T$ also satisfies both (ii) and (iii).)

**Theorem 7 (Tutte [18])** *Let $G$ be a graph. Then $G$ has a 2-factor if and only if for any $S, T \subset V(G)$ with $S \cap T = \emptyset$, $2|S| - 2|T| + d_{G-S}(T) - h_G(S,T) \geq 0$.*

**Theorem 8 (Tutte [18])** *Let $G$ be a graph. Then* (i) *$G$ has a 3-factor if and only if for any $S, T \subset V(G)$ with $S \cap T = \emptyset$, $3|S| - 3|T| + d_{G-S}(T) - h'_G(S,T) \geq 0$.* (ii) *For any $S, T \subset V(G)$ with $S \cap T = \emptyset$, $3|S| - 3|T| + d_{G-S}(T) - h'_G(S,T) \equiv |G|$ (mod 2).*

**Lemma 9 (Aldred, Egawa, Fujisawa, Ota and Saito [1])** *Let $G$ be a graph having no 2-factor. Then there exist $S, T \subset V(G)$ with $S \cap T = \emptyset$ such that* (i) $2|S| - 2|T| + d_{G-S}(T) - h_G(S,T) \leq -2$, (ii) $|S| \leq |T| - 1$, *and* (iii) *for any odd component $C$ of $G - (S \cup T)$, $e(C,T) = |N_G(C) \cap T|$.*

## 4 Proof of Theorem 1

Suppose that $G$ has no 2-factor. By Lemma 9, there exist $S, T \subset V(G)$ with $S \cap T = \emptyset$ satisfying (i)–(iii). Since $\delta(G) \geq 5$, we obtain $d_{G-S}(T) \geq 5|T| - e_G(S,T)$. Then it follows from (i) that

$$2|S| + 3|T| - e_G(S,T) - h_G(S,T) \leq -2. \tag{1}$$

**Claim 1** *For any odd component $C$ of $G - (S \cup T)$, we have $|N_G(C) \cap S| \geq 3$ or $|N_G(C) \cap T| \geq 3$.*

**Proof.** Suppose that $|N_G(C) \cap S| \leq 2$. Since $G$ is 4-connected, $|N_G(C)| \geq 4$, and hence $|N_G(C) \cap T| \geq 2$. It follows from the definition of odd component $C$ and Lemma 9 (iii) that $|N_G(C) \cap T| \geq 3$. $\square$

Let us contract each odd component $C$ to a single vertex $v_C$ and delete each component of $G - (S \cup T)$ which is not odd. Let $H$ be the resulting graph. Let

$$\begin{aligned} \tilde{S} &:= S \cup \{v_C : C \text{ is an odd component with } |N_G(C) \cap T| \geq 3\} \\ \text{and} \quad \tilde{T} &:= T \cup \{v_C : C \text{ is an odd component with } |N_G(C) \cap T| \leq 2\}. \end{aligned}$$

By Claim 1, for any $v_C \in \tilde{T} - T$, $|N_G(C) \cap S| \geq 3$. We construct the bipartite graph $H_0$ from $H$ with bipartition $\tilde{S}$ and $\tilde{T}$ by removing edges between two vertices in the same partite set. Since $H_0$ can be also embedded in a surface $\Sigma$ in which $G$ is embedded, it follows from Lemma 4 (i) and the definition of $\tilde{S}$ and $\tilde{T}$ that

$$\begin{aligned} e_G(S,T) + 3h_G(S,T) &\leq |E(H_0)| \\ &\leq 2|V(H_0)| + 2g - 4 \\ &= 2\big(|S| + |T| + h_G(S,T)\big) + 2g - 4 \end{aligned}$$

$$\text{or} \qquad e_G(S,T) \leq 2|S| + 2|T| - h_G(S,T) + 2g - 4. \tag{2}$$

By the inequalities (1) and (2), we obtain $|T| \leq 2g - 6$, and hence it follows from Lemma 9 (ii) that

$$|S \cup T| \leq 4g - 13. \tag{3}$$

Suppose that for any component $C$ of $G - \big(S \cup T\big)$, $G - C$ is not flat. By Lemma 3, there exists a non-contractible curve $\gamma$ such that $\gamma \cap G \subset S \cup T$. However, it contradicts the inequality

(3) and the fact that the face-width of $G$ is at least $4g-12$. Therefore there exists a component $C_0$ of $G-\big(S\cup T\big)$ such that $G-C_0$ is flat. We consider the graph $\tilde{H}:=H_0-\{v_{C_0}\}$ if $C_0$ is an odd component; otherwise let $\tilde{H}:=H_0$. By the choice of $C_0$, $H_0$ is flat, and hence it follows from Lemma 4 (ii) and the definition of $\tilde{S}$ and $\tilde{T}$ that

$$\begin{aligned} e_G(S,T)+3\big(h_G(S,T)-1\big) &\leq |E(\tilde{H})| \\ &\leq 2|V(\tilde{H})|-2 \\ &\leq 2\big(|S|+|T|+h_G(S,T)\big)-2 \end{aligned}$$

$$\text{or} \qquad e_G(S,T)\leq 2|S|+2|T|-h_G(S,T)+1. \tag{4}$$

By the inequalities (1) and (4), $|T|\leq -1$, a contradiction. This proves Theorem 1. □

## 5 Proof of Theorem 2

Suppose that $G$ has no 3-factor. By Theorem 8 (i) and (ii) and the fact $|G|$ is even, there exist $S,T\subset V(G)$ with $S\cap T=\emptyset$ satisfying $3|S|-3|T|+d_{G-S}(T)-h'_G(S,T)\leq -2$. Let $\omega$ be the number of components of $G-(S\cup T)$. Note that $\omega\geq h'_G(S,T)$, and hence $3|S|-3|T|+d_{G-S}(T)-\omega\leq -2$. Since $G$ is 5-connected, we obtain $d_{G-S}(T)\geq 5|T|-e_G(S,T)$. Therefore

$$3|S|+2|T|-e_G(S,T)-\omega\leq -2. \tag{5}$$

Note that for any component $C$ of $G-\big(S\cup T\big)$, $|N_G(C)\cap S|\geq 3$ or $|N_G(C)\cap T|\geq 3$, because $G$ is 5-connected.

Let us contract each component $C$ of $G-(S\cup T)$ to a single vertex $v_C$, and let $H$ be the resulting graph. Let

$$\begin{aligned} \tilde{S} &:= S\cup\{v_C : C \text{ is a component of } G-(S\cup T) \text{ with } |N_G(C)\cap T|\geq 3\} \\ \text{and} \qquad \tilde{T} &:= T\cup\{v_C : C \text{ is a component of } G-(S\cup T) \text{ with } |N_G(C)\cap T|\leq 2\}. \end{aligned}$$

Note that for any $v_C\in\tilde{T}-T$, $|N_G(C)\cap S|\geq 3$. We construct the bipartite graph $H_0$ from $H$ with bipartition $\tilde{S}$ and $\tilde{T}$ by removing edges between two vertices in the same partite set. Since $H_0$ can be also embedded in a surface $\Sigma$ in which $G$ is embedded, it follows from Lemma 4 (i) and the definition of $\tilde{S}$ and $\tilde{T}$ that

$$\begin{aligned} e_G(S,T)+3\omega &\leq |E(H_0)| \\ &\leq 2|V(H_0)|+2g-4 \\ &= 2\big(|S|+|T|+\omega\big)+2g-4 \end{aligned}$$

$$\text{or} \qquad e_G(S,T)\leq 2|S|+2|T|-\omega+2g-4. \tag{6}$$

By the inequalities (5) and (6),

$$|S|\leq 2g-6. \tag{7}$$

**Claim 2** $|T| \le 42g - 112$.

*Proof.* Let $T_{\ge 2} := \{x \in T : |N_G(x) \cap S| \ge 2\}$. Now we construct the new graph $R$ as follows; We first delete all vertices in $V(G) - (S \cup T_{\ge 2})$ and all edges connecting two vertices of $S$. For each vertex $x \in T_{\ge 2}$ in turn, we choose an arbitrary vertex $y \in N_G(x) \cap S$ and contract $x$ onto $y$. Thus $V(R) = S$. Observe that $R$ may have multiple edges. Note that $|E(R)| = \sum_{x \in T_{\ge 2}} \big(|N_G(x) \cap S| - 1\big)$.

If $R$ has three multiple edges, then it corresponds to a subgraph of $G$ isomorphic to $K_{2,3}$, which contradicts Lemma 6. Thus, the multiplicity of $R$ is at most two. Let $R'$ be the graph obtained from $R$ by removing multiple edges. Then $2|E(R')| \ge |E(R)|$. By Lemma 5, we have $|E(R')| \le 3|V(R')| + 3g - 6 = 3|S| + 3g - 6$. These imply that

$$\begin{aligned} \sum_{x \in T_{\ge 2}} \big(|N_G(x) \cap S| - 1\big) &= |E(R)| \\ &\le 2|E(R')| \\ &\le 6|S| + 6g - 12. \end{aligned}$$

Therefore

$$\begin{aligned} e_G(S,T) &= \sum_{x \in T} |N_G(x) \cap S| \\ &= |T| + \sum_{x \in T_{\ge 2}} \big(|N_G(x) \cap S| - 1\big) \\ &\le |T| + 6|S| + 6g - 12. \end{aligned}$$

Then by the inequality (5), we have

$$|T| - 3|S| - \omega \le 6g - 14. \tag{8}$$

Consider the graph $K$ obtained from $G$ by contacting each component $C$ of $G - (S \cup T)$ to a single vertex, and deleting any edge in $S \cup T$. Note that $K$ is a bipartite graph with bipartition $S \cup T$ and the set of components of $G - (S \cup T)$. Then by Lemma 4, $|E(K)| \le 2|V(K)| + 2g - 4 = 2|S| + 2|T| + 2\omega + 2g - 4$. Since $G$ is 5-connected, we obtain $|E(K)| \ge 5\omega$, and hence

$$\begin{aligned} & 5\omega \le 2|S| + 2|T| + 2\omega + 2g - 4 \\ \text{or} \quad & \omega \le \frac{1}{3}\big(2|S| + 2|T| + 2g - 4\big). \end{aligned} \tag{9}$$

By the inequalities (7)–(9), we have

$$\begin{aligned} |T| &\le 11|S| + 20g - 46 \\ &\le 42g - 112. \qquad \square \end{aligned}$$

By the inequality (7) and by Claim 2, we have $|S \cup T| \le 44g - 118$. Suppose that for any component $C$ of $G - (S \cup T)$, $G - C$ is not flat. By Lemma 3, there exists a non-contractible curve $\gamma$ such that $\gamma \cap G \subset S \cup T$. However, it contradicts the fact the face-width of $G$ is at least $44g - 117$. Therefore there exists a component $C_0$ of $G - (S \cup T)$ such that $G - C_0$ is flat. Consider the graph

$\tilde{H} := H_0 - \{v_{C_0}\}$. By the choice of $C_0$, $H_0$ is a plane graph, and hence it follows again from Lemma 4 (ii) and the definition of $\tilde{S}$ and $\tilde{T}$ that

$$\begin{aligned} e_G(S,T) + 3(\omega - 1) &\leq |E(H_0)| \\ &\leq 2|V(H_0)| - 2 \\ &= 2\big(|S| + |T| + \omega - 1\big) - 2 \end{aligned}$$

$$\text{or} \qquad e_G(S,T) \leq 2|S| + 2|T| - \omega - 1. \tag{10}$$

By the inequalities (5) and (10), $|S| \leq -3$, a contradiction. This proves Theorem 2. □

## 6 Concluding Remarks

In this section, we show that the conditions "$\delta(G) \geq 5$" of Theorem 1 and 5-connectedness of Theorem 2 are best possible in a sense. Actually, for any surface $\Sigma$ of Euler genus $g \geq 3$, there exist infinitely many 4-connected graphs on $\Sigma$ with sufficiently large face-width having no 2-factor nor 3-factor. Let $H$ be a quadrangulation of $\Sigma$ with sufficiently large face-width. Construct the graph $G$ from $H$ by adding a vertex into each face of $H$ and let $S := V(H)$ and $T := V(G) - S$. By the construction of $G$, $|T| = |F(H)| = |S| + g - 2$ and $d_{G-S}(T) = h_G(S,T) = h'_G(S,T) = 0$. This implies that $2|S| - 2|T| + d_{G-S}(T) - h_G(S,T) = 4 - 2g < 0$, and $3|S| - 3|T| + d_{G-S}(T) - h'_G(S,T) = 6 - 3g < 0$, because $g \geq 3$. Hence it follows from Theorems 7 and 8 (i) that $G$ has no 2-factor nor 3-factor.

Finally, a few words on the face-width bound of Theorems 1 and 2. Although we cannot construct an example which shows that the face-width in Theorems 1 and 2 must depend on Euler genus, we suspect that we can modify the example due to Archdeacon, Hartsfield, and Little [2]. For each $k$, they constructed a $k$-connected triangulation of an orientable surface having face-width $k$ in which every spanning tree has a vertex of degree at least $k$.

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