A simple algorithm for 4-coloring 3-colorable planar graphs

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ABSTRACT

Graph coloring for 3-colorable graphs receives very much attention by many researchers in theoretical computer science. Deciding 3-colorability of a graph is a well-known NP-complete problem. So far, the best known polynomial approximation algorithm achieves a factor of $O(n^{0.2072})$, and there is a strong evidence that there would be no polynomial time algorithm to color 3-colorable graphs using at most c colors for an absolute constant c.

In this paper, we consider 3-colorable PLANAR graphs. The Four Color Theorem (4CT) [1, 2, 14] gives an $O(n^2)$ time algorithm to 4-color any planar graph. However the current known proof for the 4CT is computer-assisted. In addition, the correctness of the proof is still lengthily and complicated.

We give a very simple $O(n^2)$ algorithm to 4-color 3-colorable planar graphs. The correctness needs only a 2-page proof.

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1 Introduction

Graph coloring is arguably the most popular subject in graph theory. Also, it is one of the central problems in combinatorial optimization, since it is one of the hardest problems to approximate. In general, the chromatic number is inapproximable in polynomial time within factor $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $coRP = NP$, cf. Feige and Kilian [8] and Håstad [9].

Graph coloring for 3-colorable graphs receives much attention by many researchers in theoretical computer science. Let us first observe that deciding 3-colorability (which we call the 3-coloring problem) is one of the well-known NP-complete problems. A natural question is: Can we get an approximation algorithm for coloring 3-colorable graphs? The first non-trivial result was due to Wigderson [15] who gave an $O(\sqrt{n})$-approximation algorithm. The factor $O(\sqrt{n})$ was improved by Blum [3] to $O(n^{3/8})$. Then Blum and Karger [4] observed that it is possible to use the method of Karger, Motwani, and Sudan [11] to improve to $O(n^{3/14})$, which is where things stood for a decade. Currently, the best known polynomial approximation algorithm achieves a factor of $O(n^{0.2072})$ [5]. For the negative side, Dinur et al. [6] gave some evidence that assuming the well-known Khot's unique game conjecture [12] (which we do not mention here, but let us just mention that the truth of this conjecture would give

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rise to inapproximable results for many NP-hard problems, including the 3-coloring problem), it is not possible to give a coloring of 3-colorable graph, using at most c colors for some absolute constant c in polynomial time. It is a well-known open problem whether or not a 3-colorable graph can be colored using at most $O(poly(logn))$ colors in polynomial time, see [6].

As we see here, the fact that an input graph is 3-colorable does not help much for graph coloring. However what about a planar graph? Of course, the Four Color Theorem (4CT) [1, 2, 14] gives an $O(n^2)$ time algorithm to 4-color any planar graph. However the current known proof for the 4CT is computer-assisted. In fact, for both the unavoidability (i.e, finding a forbidden subgraph in a minimal counterexample) and the reducibility (i.e, proving that a certain configuration would not appear in a minimal counterexample) parts of the known proof, we have to use computer. Even the computer-free part is not so simple.

Our motivation is that if we restrict our attention to 3-colorable planar graphs, we may be able to find a very simple algorithm and a very easy correctness proof. In this paper, we shall prove that this is, indeed, the case. Specifically,

Theorem 1 *Let G be a 3-colorable planar graph. Then there exists an $O(n^2)$-time algorithm to find a 4-coloring of G, where $n = |G|$.*

Moreover, the correctness of the proof only requires a 2-page (and of course, it is computer-free). This is a big contrast with the 4CT. In addition, our algorithm may be used to obtain the following.

Corollary 2 *Given a planar graph G, there is an $O(n^2)$-time algorithm to either output a 4-coloring of G or conclude that G is not 3-colorable.*

Thus this corollary tells us that the difficulty in the 4CT is the non-3-colorable case. Let us point out that 3-colorability of planar graphs is a well-known NP-complete problem. Thus we cannot expect to have a complete characterization of non-3-colorable planar graphs.

Our proof of Theorem 1 uses some basic idea for 4-coloring planar graphs, but when we apply our algorithm recursively, we have to be a little careful because we have to preserve the 3-colorability of the current graph. In the proof of the 4CT, there are many places that some subgraph is contracted. We may not use this, because this may destroy the 3-colorablity. Deleting a vertex is always usable. We also introduce another reduction. Namely, we sometimes identify two vertices u, v in a same face to get the new graph G'. The important point is that we can show that the original graph G is 3-colorable if and only if G' is 3-colorable. Details are explained in the next section.

2 Proof and Algorithm

Before proving our main theorem, we need some definitions. For the basic notation, we refer to the book [13]. Let G be a plane graph. We define $V(G)$, $F(G)$ by the set of vertices of G and the set of faces of G, respectively. We also define G^* by the dual graph of G. Let v be a vertex of G, and let f be a face of G. Let us denote degree of v in G by $d_G(v)$. Since G is a plane graph and the size of the face f is exactly the same as the degree of f in the dual graph G^*, thus we can define $d_{G^*}(f)$ as the size of the face f. Note that the dual graph G^* may have multiple edges or loops, but we allow them and consider the degree in G^* containing them. (For a loop of G^*, we count it twice.) A quadrangle is a face of size 4 and a pentagon is a face of size 5. We need to define some special configurations. For a simple plane graph G, we call a vertex v of degree five *bad* if all faces incident with v, except for at most one, are triangles and the exceptional face has size at most five. Moreover, v is Type I, Type II and Type III if the exceptional face is a triangle, a quadrangle, and a pentagon, respectively. See Figure 1. We have the following restrictions for a 3-coloring around a bad vertex.

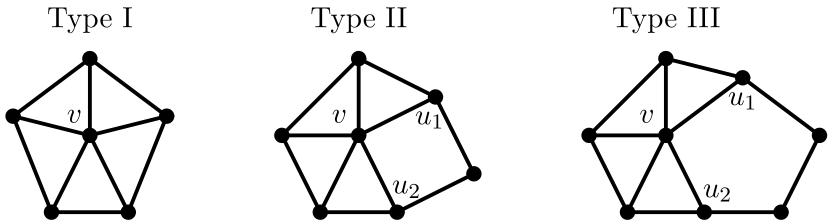

Figure 1: A bad vertex.

Fact 3 *If a plane graph G has a bad vertex of Type I, then G is not 3-colorable.*

Fact 4 *Suppose that a plane graph G has a bad vertex v of Type II or Type III, and G is 3-colorable. Then u_1 and u_2 are contained in the same color class for any 3-coloring of G, where u_1 and u_2 are two neighbors of v which are contained in a quadrangle or pentagon. In other words, G is 3-colorable if and only if G' is, where G' is obtained from G by identifying u_1 and u_2 (and deleting the resulting multiple edges).*

The proofs of the above facts are easy, so we omit them.

We now prove the key lemma.

Lemma 5 *Every simple planar graph G contains (i) a vertex of degree at most 4 or (ii) a bad vertex.*

Proof. We will show this lemma by discharging methods.

We define the initial charge function w for each $V(G) \cup F(G)$ as $w(v) := d_G(v) - 6$ for $v \in V(G)$ and $w(f) := 2d_{G^*}(f) - 6$ for $f \in F(G)$. By Euler's Formula, we have $\sum_{v \in V(G)} w(v) + \sum_{f \in F(G)} w(f) = -12$. The new function w^* is obtained by the following discharging rules;

(R1) Send $\frac{1}{2}$ from each quadrangle or pentagon to each of incident vertex of degree five.

(R2) Send 1 from each face of size at least 6 to each of incident vertex of degree five.

We will show that if G has no bad vertex nor a vertex of degree at most 4, then $w^*(v) \geq 0$ for any $v \in V(G)$ and $w^*(f) \geq 0$ for any $f \in F(G)$. This would contradict that $\sum_{v \in V(G)} w^*(v) + \sum_{f \in F(G)} w^*(f) = \sum_{v \in V(G)} w(v) + \sum_{f \in F(G)} w(f) = -12$.

Let $f \in F(G)$. Since G is simple, $d_{G^*}(f) \geq 3$ for any $f \in F(G)$. Then we have

$$w^*(f) \geq \begin{cases} w(f) = 0 & \text{if } d_{G^*}(f) = 3, \\ w(f) - 4 \cdot \frac{1}{2} = 0 & \text{if } d_{G^*}(f) = 4, \\ w(f) - 5 \cdot \frac{1}{2} = \frac{3}{2} & \text{if } d_{G^*}(f) = 5, \\ w(f) - d_{G^*}(f) \geq 0 & \text{otherwise.} \end{cases}$$

Thus for each face f, $w^*(f) \geq 0$.

It remains to prove that $w^*(v) \geq 0$ for every vertex v. Take a vertex $v \in V(G)$. We may assume that $d_G(v) \geq 5$, or there is a vertex of degree at most four (in this case, we are done). If $d_G(v) \geq 6$, then $w^*(v) = w(v) \geq 0$. Thus, we may assume that $d_G(v) = 5$. If v is incident with a face of size at least 6, then $w^*(v) \geq w(v) + 1 = 0$. This implies that v is only incident with a face of size at most

5. So, if G has no bad vertex, then v must be incident with at least two faces of size 4 or 5, In either case, we have $w^*(v) \geq 0$, and this completes the proof of Lemma 5. □

The following fact is already known (see [13] for example), but for the completeness, we shall include the proof.

Fact 6 *Let G be a plane graph with a 4-coloring c' and let f be a face of size at least four. Take four vertices x_1, x_2, x_3, x_4 (along clockwise order) in f. Then G also has a 4-coloring such that at most three colors are used for x_1, x_2, x_3, x_4. Moreover, given the graph G and the coloring c', we can find such a 4-coloring in $O(n)$ time, where $n = |G|$.*

Proof. Suppose that a 4-coloring c' of G uses 4 colors for x_1, x_2, x_3, x_4, say x_i is colored by the color i. Let us recall that an (i,j)-Kempe chain containing a vertex v is the component induced by the color classes i and j that contains the vertex v. Consider the $(1,3)$-Kempe chain containing x_1. If it does not contain x_3, then by changing colors 1 and 3 in the Kempe chain, we can color x_1 by the color 3, which gives rise to a desired coloring. If it contains x_3, then by the planarity, the $(2,4)$-Kempe chain containing x_2 does not contain x_4, and hence by changing colors 2 and 4 in the $(2,4)$-Kempe chain containing x_2, we can color x_2 by the color 4, which gives rise to a desired 4-coloring.

Clearly, given the graph G and the coloring c', this change takes only $O(n)$ time. □

We are now ready to prove Theorem 1.

Proof of Theorem 1.

Given a plane graph G with n vertices, which is 3-colorable, we have to give a 4-coloring. We may assume that G is simple. We use the following algorithm recursively. The algorithm proceeds as follows:

Step 1. If the current graph has at most 3 vertices, we just give a 3-coloring. Otherwise, delete a vertex of degree at most 4 in the current graph G.

Step 2. Apply the linear time algorithm by Eppstein [7] to find a bad vertex v.

By Lemma 5, G has (i) a vertex of degree at most 4 or (ii) a bad vertex. At the moment, there is no vertex of degree at most 4. Thus we may assume that there is such a bad vertex v. By Fact 3, v is not Type I, so is Type II or III. The linear time algorithm of Eppstein [7] will find a subgraph isomorphic to one of two graphs (Type II and Type III) drawn in Figure 1.

We then construct the new graph G' by identifying two vertices u_1 and u_2, as in Fact 4. It follows from Fact 4 that G' is also 3-colorable. Then rerun the algorithm on G'.

Step 3. Extend the coloring c' of the current graph to the original graph G (by possibly changing the coloring of c') in $O(n)$ time.

At the moment, we have a 4-coloring c of the current graph G'. We need to extend the c to the original input graph G (by possibly changing the coloring of c'). In Step 2, we identify two vertices u_1, u_2. The reverse operation clearly extends the coloring c. In Step 1, we delete a vertex v of degree at most 4. We now need to put a vertex v back to G' to obtain the resulting graph G. If $d_G(v) \leq 3$, then the coloring c can be easily extended for v. If $d_G(v) = 4$, then we can also change the coloring c' of G' to color v by Fact 6. All of these processes can be done in $O(n)$ time by Fact 6.

Both steps 1, 2 and 3 can be implemented in linear time. Another factor of n pops up because of applying the recursion. Hence we can find a 4-coloring in $O(n^2)$ time. □

In the above algorithm, a plane graph G is given for the input. However, even if the embedding of G is not given, we can use the linear time algorithm for the planarity testing [10] to get a plane embedding of G. Then we can apply Theorem 1 to this plane graph.

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