Non-separating subgraphs after deleting many disjoint paths

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ABSTRACT

Motivated by the well-known conjecture by Lovász [6] on the connectivity after the path removal, we study the following problem:

There exists a function $f = f(k, l)$ such that the following holds. For every $f(k, l)$-connected graph G and two distinct vertices s and t in G, there are k internally disjoint paths $P_1, \ldots, P_k$ with endpoints s and t such that $G - \bigcup_{i=1}^{k} V(P_i)$ is l-connected.

When $k = 1$, this problem corresponds to Lovász conjecture, and it is open for all the cases $l \geq 3$.

We show that $f(k, 1) = 2k + 1$ and $f(k, 2) \leq 3k + 2$. The connectivity "$2k + 1$" for $f(k, 1)$ is best possible. Thus our result generalizes the result by Tutte [8] for the case $k = 1$ and $l = 1$ (the first settled case of Lovász conjecture), and the result by Chen, Gould and Yu [1], Kriesell [4], Kawarabayashi, Lee, and Yu [2], independently, for the case $k = 1$ and $l = 2$ (the second settled case of Lovász conjecture).

When $l = 1$, our result also improves the connectivity bound "$22k + 2$" given by Chen, Gould and Yu [1].

June 2009, revised October 11, 2010.

Keywords: Non-separating subgraphs and Lovász conjecture.

1 Introduction

The following well-known conjecture is due to Lovász [6]:

Conjecture 1 *There exists a function $g = g(l)$ such that the following holds. For every $g(l)$-connected graph G and two distinct vertices s and t in G, there exists a path P with endpoints s and t such that $G - V(P)$ is l-connected.*

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Conjecture 1 can also be phrased in terms of finding a cycle containing an arbitrary edge e such that deleting the vertices of the cycle leaves the graph l-connected. At the same time, Lovász conjectured that every $(l+3)$-connected graph G contains a cycle C such that $G-V(C)$ is l-connected. This was proved by Thomassen [7]. Conjecture 1 is known to be true in several small cases. A path P connecting two vertices s and t in a given graph G such that $G-V(P)$ is connected, is called a *non-separating path*. It follows from a famous result of Tutte [8] that any 3-connected graph contains a non-separating path connecting any two vertices, and consequently, $g(1)=3$. The case $l=2$ was independently obtained by [1] and [4], who showed $g(2)=5$. In fact, Kawarabayashi, Lee and Yu [2] have characterized all 4-connected graphs that have two vertices s and t such that there is no path P with endpoints s and t so that $G-V(P)$ is 2-connected. But as far as we are aware, Conjecture 1 is still (wide) open for $l \geq 3$, and the prospect is not bright (although a weaker version of Lovász' conjecture was solved in [3], which settles a conjecture by Kriesell [5].).

In [1], the authors also show that in a $(22k+2)$-connected graph, there exist k internally disjoint non-separating paths $P_1,\dots,P_k$ connecting any pair of vertices. In fact, they also proved that $G-\bigcup_{i=1}^{k} V(P_i)$ is connected. A related result is given in [9]. These results motivate us to propose the following conjecture:

Conjecture 2 *There exists a function $f=f(k,l)$ such that the following holds. For every $f(k,l)$-connected graph G and two distinct vertices s and t in G, there are k internally disjoint paths $P_1,\dots,P_k$ with endpoints s and t such that $G-\bigcup_{i=1}^{k} V(P_i)$ is l-connected.*

Note that when $k=1$, Conjecture 2 is exactly Lovász conjecture. In this paper, we improve the above mentioned connectivity result by Chen, Gould and Yu [1], and give the best possible connectivity bound. Namely:

Theorem 3 *Let k be an integer with $k \geq 1$, let G be a $(2k+1)$-connected graph and let $s,t \in V(G)$ with $s \neq t$. Then there exist k internally disjoint paths $P_1,P_2,\dots,P_k$ with endpoints s and t such that $G-\bigcup_{i=1}^{k} V(P_i)$ is connected.*

Let us observe that Theorem 3 is a far generalization of the Tutte's above mentioned result which corresponds to the case $l=1$ in Conjecture 1.

Note that the following graph shows that the connectivity condition on Theorem 3 is best possible. Let $A \cup B$ be a clique of size $4k$ with $|A|=|B|=2k$. For each $2k$ vertices W of $A \cup B$ such that half of them belong to A and the other half belong to B, we add $k+2$ vertices such that each vertex is adjacent to all the vertices in W. Thus we add $(k+2)\binom{2k}{k}\binom{2k}{k}$ vertices. Finally we add vertices s and t such that s is adjacent to all the vertices of A and t is adjacent to all the vertices of B, and we call the resulting graph G. Note that G is $2k$-connected. Whenever we take k pairwise internally disjoint paths $P_1,P_2,\dots,P_k$ from s to t, $\bigcup_{i=1}^{k} P_i$ must use at least k vertices of A and at least k vertices of B. Now we consider the added $(k+2)$ vertices which are joined to such $2k$ vertices. Since at most k of them can be on one of the paths $P_1,P_2,\dots,P_k$, at least two vertices are not used, and hence $G-\bigcup_{i=1}^{k} V(P_i)$ is not connected.

We actually prove the following stronger result, whose proof also gives Theorem 3.

Theorem 4 *Let k be an integer with $k \geq 1$, let G be a $(3k+2)$-connected graph and let $s, t \in V(G)$ with $s \neq t$. Then there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ with endpoints s and t such that $G - \bigcup_{i=1}^{k} V(P_i)$ is 2-connected.*

Let us observe that Theorem 4 is a far generalization of the above mentioned result [1, 2, 4] which corresponds to the case $l = 2$ in Conjecture 1. In fact, when $k = 1$ in Theorem 4, Theorem 4 implies the above mentioned result [1, 2, 4]. But we do not know if the connectivity "$3k+2$" is best possible (except for the case $k = 1$, which is best possible, as demonstrated in [2]). We can easily modify the above mentioned example which shows that $f(k, 2) \geq 2k + 2$, but we do not know if this is the lower bound for the connectivity for $f(k, 2)$.

Before we prove Theorems 3 and 4, we give some notations.

A block of a graph G is a maximal connected subgraph of G that has no cut vertex. Note that any block of a connected graph of order at least two is 2-connected or isomorphic to K_2.

For a path P and for two vertices $u, v \in V(P)$ (possibly $u = v$), we denote the subpath of P from u to v by $P[u, v]$. Note that $P[u, v] = P[v, u]$. Let $P(u, v] := P[u, v] - \{u\}$, $P[u, v) := P[u, v] - \{v\}$, and $P(u, v) := P[u, v] - \{u, v\}$. For convenience, $P(u, u] = P[u, u) = P(u, u) = \emptyset$. Let P_1, P_2 be two paths with end vertices s_1 and s_2, respectively. For two vertices u_1 and u_2 with $u_i \in V(P_i)$ for $i = 1, 2$ and $u_1u_2 \in E(G)$, we denote the path from s_1 to s_2 obtained by combining P_1 and P_2 using the edge u_1u_2 by $P_1[s_1, u_1]P_2[u_2, s_2]$.

In the proof of our main theorem, we use lexicographic order. For two sequences $(a_1, \ldots, a_l)$ and $(b_1, \ldots, b_{l'})$ with $l < l'$ and $a_i = b_i$ for any $1 \leq i \leq l$, we regard that $(b_1, \ldots, b_{l'})$ is larger than $(a_1, \ldots, a_l)$ in lexicographic order.

2 Proof of Theorems

As we said before, our proof of Theorem 4 will give Theorem 3 too. Thus we first give a proof of Theorem 4.

Proof of Theorem 4.

Since G is $(3k+2)$-connected, there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ with endpoints s and t in G such that $|G'| \geq 3$ and G' has at least one edge, where $G' := G - \bigcup_{i=1}^{k} V(P_i)$ (by just finding an edge e whose endpoints are not any of s and t, and then finding k disjoint paths between s and t in $G - e$). Let R be the maximum block in G' and let l be the number of components of $G' - R$. If $l = 0$, then $R = G'$ is 2-connected since $|G'| \geq 3$, and hence there is nothing to prove. So we may assume that $l \geq 1$. Let $H_1, H_2, \ldots, H_l$ be components of $G' - R$ with $|H_1| \geq |H_2| \geq \cdots \geq |H_l|$. Take such k internally disjoint paths $P_1, P_2, \ldots, P_k$ so that

(P1) $|R|$ is as large as possible,

(P2) $\big(|H_1|, |H_2|, \ldots, |H_l|\big)$ is as large as possible in lexicographic order, subject to (P1).

By (P1) and (P2), we obtain the following claim.

Claim 1 *For any $1 \leq r \leq k$, there exist no $2r$ vertices $u_1, v_1, \ldots, u_r, v_r$ such that $u_i, v_i \in V(P_i)$, $u_i \in V\big(P_i[s, v_i)\big)$, $u_i v_{i+1} \in E(G)$ (the index is taken modulo r), and $\bigcup_{i=1}^{r} V\big(P_i(u_i, v_i)\big) \neq \emptyset$. In particular, P_i has no chords.*

Proof. Suppose not. Let $Q_i := P_i[s, u_i]P_{i+1}[v_{i+1}, t]$ (the index is taken modulo r) for $1 \leq i \leq r$ and let $Q_i := P_i$ for $r+1 \leq i \leq k$. Then $Q_1, Q_2, \ldots, Q_k$ are k internally disjoint paths with endpoints s and t and fewer vertices than $P_1, P_2, \ldots, P_k$. This contradicts (P1) or (P2). □

Note that when we apply Claim 1, we may reorder the paths $P_1, \ldots, P_k$ if necessary.

We say that each path P_i goes from left (closer to s) to right (closer to t). Let a_i be the leftmost neighbor of H_l in P_i and b_i be the rightmost neighbor if $N_G(H_l) \cap V(P_i) \neq \emptyset$. Now we will perform the following operation, and we shall update the vertices a_i and b_i for some $1 \leq i \leq k$ at each step.

Operation 1: We define

$$\bar{P}_i := \begin{cases} P_i[s, a_i) \cup P_i(b_i, t] & \text{if } a_i \text{ and } b_i \text{ exist,} \\ P_i & \text{otherwise.} \end{cases}$$

Suppose that there exists an edge uv connecting $\bar{P}_i$ and $P_j(a_j, b_j)$ for some $1 \leq i, j \leq k$ with $u \in V(\bar{P}_i)$ and $v \in V(P_j(a_j, b_j))$. (Note that $i \neq j$ by Claim 1.) If $u \in V(P_i[s, a_i))$, then we regard u as the new a_i, and if $u \in V(P_i(b_i, t])$, then we regard u as the new b_i. If a_i and b_i do not exist, then we let $a_i = b_i = u$ as the new a_i and b_i.

We perform Operation 1 as many times as possible. In each step, $\bigcup_{i=1}^{k} V(\bar{P}_i)$ becomes smaller, and hence it must stop. Note that $a_i = b_i$ could happen. For the last a_i's and b_i's, let $A := \{a_i : 1 \leq i \leq k \text{ and } a_i \text{ exists}\}$ and $B := \{b_i : 1 \leq i \leq k \text{ and } b_i \text{ exists}\}$. For convenience, let $P_i(a_i, b_i) = \emptyset$ if a_i and b_i do not exist. Note that a_i could be s and b_i could be t.

By the definition of A and B, we obtain the following claim.

Claim 2 *There exists no edge connecting $\bigcup_{i=1}^{k} \bar{P}_i$ and $\bigcup_{i=1}^{k} P_i(a_i, b_i) \cup H_l$.*

Proof. Suppose that there exists an edge uv connecting $\bigcup_{i=1}^{k} \bar{P}_i$ and $\bigcup_{i=1}^{k} P_i(a_i, b_i) \cup H_l$, say $u \in V(\bar{P}_i)$ and $v \in V(P_j(a_j, b_j)) \cup H_l$ for some $1 \leq i, j \leq k$. If $v \in H_l$, this contradicts the first choice of a_i or b_i, and if $i = j$, this contradicts Claim 1. Thus $v \in V(P_j(a_j, b_j))$ and $i \neq j$. However, we can still perform Operation 1 for uv, a contradiction again for our final choice A, B. □

Moreover, the construction of a_i and b_i implies the following claim, which is crucial for our proof.

Claim 3 *For each $1 \leq i \leq k$, there exist k internally disjoint paths $Q_1, Q_2, \ldots, Q_k$ with endpoints s and t in $\big(\bigcup_{j=1}^{k} V(P_j) - P_i(a_i, b_i)\big) \cup H_l$.*

Proof. By the symmetry, we only need to prove the case $i = 1$. If $P_1(a_1, b_1) = \emptyset$, then $P_1, \ldots, P_k$ satisfy the desired condition, so there is nothing to prove. Thus, we may assume that $P_1(a_1, b_1) \neq \emptyset$, and hence a_1 and b_1 exist with $a_1 \neq b_1$.

For a vertex $u \in \bigcup_{i=1}^{k} V(P_i) - \{s,t\}$, let $\tau(u)$ be the integer with $u \in V(P_{\tau(u)})$. We will define the sequence of vertices $v_1, u_1, v_2, u_2, v_3, \ldots v_{p+1}$ as follows. Let v_1 be an arbitrary vertex in $P_1(a_1, b_1)$ and let $u_1 := a_1$. For $p \geq 1$, if $N_G(u_p) \cap H_l \neq \emptyset$, then let v_{p+1} be any neighbor of u_p in H_l. Otherwise, by the definition of Operation 1, u_p was chosen as a neighbor of some vertex $v_{p+1} \in V\big(P_{\tau(v_{p+1})}(a_{\tau(v_{p+1})}, b_{\tau(v_{p+1})})\big)$ and let $u_{p+1} := a_{\tau(v_{p+1})}$. Until $v_{m+1} \in H_l$ for some m, we successive define the vertices u_p's and v_p's. Note that by Operation 1, $u_p \neq v_p$ and $v_p \notin A \cup B$ for any p. The following subclaim guarantees that the above sequence of vertices $v_1, u_1, v_2, u_2, v_3, \ldots, v_m, u_m, v_{m+1}$ are well-defined, and moreover, $m \leq k$.

Subclaim 1 *For any p, p' with $p \neq p'$, $\tau(v_p) \neq \tau(v_{p'})$.*

Proof. Assume that there exist two vertices v_p and $v_{p'}$ with $\tau(v_p) = \tau(v_{p'})$ and $p < p'$. Note that $u_p = u_{p'}$. Let $r := p' - p$ and we choose such p and p' so that r is as small as possible. By the minimality of r, $\tau(v_j) \neq \tau(v_{j'})$ for any $p < j < j' \leq p'$. Now we have $2r$ vertices $v_{p+1}, u_{p+1} \ldots, u_{p'-1}, v_{p'}, u_{p'}$ such that $v_j, u_j \in V(P_{\tau(v_j)})$, $u_j \in V(P_{\tau(v_j)}[s, v_j))$ and $u_{j-1}v_j \in E(G)$ for $p+1 \leq j \leq p'$.

Let j'' be an integer with $p+1 \leq j'' \leq p'$ such that $u_{j''}$ was the earliest vertex that is chosen as a vertex in A among $u_{p+1}, \ldots, u_{p'-1}, u_{p'}$ in Operation 1. By this choice, $u_{j''+1}$ (or u_{p+1} when $j'' = p'$) was not chosen yet when we chose $u_{j''}$. This implies that there exists at least one vertex in $P_{\tau(v_{j''+1})}(u_{j''+1}, v_{j''+1})$, which corresponds to the older $a_{\tau(v_{j''+1})}$ when we chose $u_{j''}$. However, this contradicts Claim 1, because $V\big(P_{\tau(v_{j''+1})}(u_{j''+1}, v_{j''+1})\big) \neq \emptyset$. □

We symmetrically define the other sequence of vertices $y_1, x_1, y_2, x_2, y_3, \ldots$ as follows. Let $y_1 = v_1$ and let $x_1 := b_1$. For $q \geq 1$, if $N_G(x_q) \cap H_l \neq \emptyset$, then let y_{q+1} be any neighbor of x_q in H_l. Otherwise, by the definition of Operation 1, x_q was chosen as a neighbor of some vertex $y_{q+1} \in V\big(P_{\tau(y_{q+1})}(a_{\tau(y_{q+1})}, b_{\tau(y_{q+1})})\big)$ and let $x_{q+1} := b_{\tau(y_{q+1})}$. Note that by Operation 1, $x_q \neq y_q$ and $y_q \notin A \cup B$ for any q. By the symmetry to Subclaim 1, we obtain the following subclaim, and hence the above sequence of vertices $y_1, x_1, y_2, x_2, y_3, \ldots, y_n, x_n, y_{n+1}$ (with $y_{n+1} \in H_l$) are well-defined, and moreover, $n \leq k$.

Subclaim 2 *For any q, q' with $q \neq q'$, $\tau(y_q) \neq \tau(y_{q'})$.*

Now we give the direction to the edges in $P_1, P_2, \ldots, P_k$ as follows. For each edge $u_p v_{p+1}$ (resp. $x_q y_{q+1}$), give the direction from u_p (resp. y_{q+1}) to v_{p+1} (resp. x_q). For the two vertices v_{m+1}, y_{n+1} with $v_{m+1}, y_{n+1} \in H_l$, let P be a path of H_l from v_{m+1} to y_{n+1}, and give the direction to the edges of P along with P from v_{m+1} to y_{n+1}. For each path P_i, we give the direction to each edge e from the left to the right, following s to t along the path P_i, except for the edges in

(I-i) $P_i[u_p, y_q]$
(I-ii) $P_i[y_q, v_p]$ if $v_p, y_q \in V(P_i)$ for some p and q and if $y_q \in V(P_i[s, v_p))$,
(I-iii) $P_i[v_p, x_q]$

(II) $P_i[u_p, v_p]$ if $u_p \in V(P_i)$ for some p and (I) does not occur,

(III) $P_i[y_q, x_q]$ if $y_q \in V(P_i)$ for some q and (I) does not occur.

For the edges in (I-i), (I-iii), (II), or (III), we give no direction and for the edges in (I-ii), we give the reverse direction, that is, from the right to the left, along the path P_i from v_p to y_q.

Note that any edge in $P_1[a_1, b_1]$ has no direction. By Subclaims 1 and 2, the above direction of edges implies that s has out-degree k and in-degree 0, t has out-degree 0 and in-degree k, and any other vertex has out-degree 1 and in-degree 1, or out-degree 0 and in-degree 0, because $v_p, y_q \notin A \cup B$ for any p and q. We now delete all the edges that have no assigned direction. We claim that there are new k pairwise internally disjoint directed paths $Q_1, Q_2, \dots, Q_k$ from s and t in $\left(\bigcup_{j=1}^{k} V(P_j) - P_1(a_1, b_1)\right) \cup H_l$. To see this, since each vertex, except for s and t, has in-degree and out-degree exactly one, thus each vertex in P_i can hit at most one directed path from s to t. This completes the proof of Claim 3. □

By Claim 3, we have the following two claims.

Claim 4 *For any H_i with $i \neq l$, there exists no edge connecting H_i and $\bigcup_{j=1}^{k} P_j(a_j, b_j)$.*

Proof. Suppose that $N(H_i) \cap V(P_j(a_j, b_j)) \neq \emptyset$ for some $i \neq l$ and for some $1 \leq j \leq k$, say $N(H_i) \cap V(P_1(a_1, b_1)) \neq \emptyset$. By Claim 3, there exist k pairwise internally disjoint paths $Q_1, Q_2, \dots, Q_k$ from s to t in $\left(\bigcup_{j=1}^{k} V(P_j) - P_1(a_1, b_1)\right) \cup H_l$. Note that for any H_r with $1 \leq r \leq l-1$, all the vertices of H_r are contained in the same component of $G - \bigcup_{j=1}^{k} V(Q_j) - R$, and $H_i \cup P_1(a_1, b_1)$ is contained in one component of $G - \bigcup_{j=1}^{k} V(Q_j) - R$. This contradicts the choice (P1) or (P2). □

Claim 5 *For any $1 \leq i \leq k$, there exists a vertex z_i in R such that there exist no edges connecting $R - \{z_i\}$ and $P_i(a_i, b_i)$.*

Proof. Suppose not. Then we can take two edges $e_1 := x_1y_1$ and $e_2 := x_2y_2$ from R to $P_i(a_i, b_i)$ such that $x_1 \neq x_2$ and $x_1, x_2 \in R$. (Possibly $y_1 = y_2$.) By Claim 3, there exist k pairwise internally disjoint paths $Q_1, Q_2, \dots, Q_k$ from s to t in $\left(\bigcup_{j=1}^{k} V(P_j) - P_i(a_i, b_i)\right) \cup H_l$. Since $x_1 \neq x_2$, we obtain $R \cup e_1 \cup P_i[y_1, y_2] \cup e_2$ is 2-connected and is contained in $G - \bigcup_{i=1}^{k} V(Q_i)$. This contradicts the choice (P1). □

For each $1 \leq i \leq k$, we take the vertex z_i as in Claim 5, and let z be a cut vertex of G' which separates R and H_l when H_l is contained in the same component of G' as R; otherwise let z be an arbitrary vertex in G'. Possibly, $z_i = z_j$ for some $1 \leq i < j \leq k$. By Claims 2, 4 and 5, $S := A \cup B \cup \{z_1, z_2, \dots, z_k, z\}$ separates $\bigcup_{i=1}^{k} P_i(a_i, b_i) \cup H_l$ from the other part. If $V(G) - S - \bigcup_{i=1}^{k} P_i(a_i, b_i) - H_l \neq \emptyset$, then S is actually a cut set, but this contradicts the connectivity condition because $|S| \leq 3k + 1$. Therefore $V(G) = S \cup \bigcup_{i=1}^{k} P_i(a_i, b_i) \cup H_l$, and this implies that $l = 1$ and $V(G') - H_l = R = \{z_1, z_2, \dots, z_k, z\}$, $A = \{s\}$ and $B = \{t\}$.

Since $|R| \geq 2$, there exists a vertex x in $R - \{z\}$, and let r be the integer such that there exist r paths P_i with $N_G(x) \cap V(P_i(s, t)) \neq \emptyset$. Take such a vertex x so that r is as small as possible. By the symmetry, we may assume that $N_G(x) \cap V(P_i(s, t)) \neq \emptyset$ for any $1 \leq i \leq r$. Note that $r \geq 1$. Since z_j has no neighbors in P_i (with $z_i \neq z_j$), our choice of x implies that $|R - \{z\}| \leq \frac{k}{r}$, and hence $|R| \leq \frac{k}{r} + 1$. Since $N_G(x) \subset \{s, t\} \cup \left(R - \{x\}\right) \cup \bigcup_{i=1}^{r} V(P_i(s, t))$ and $d_G(x) \geq 3k + 2$, we obtain $|N_G(x) \cap \bigcup_{i=1}^{r} V(P_i(s, t))| \geq (3k + 2) - 2 - \frac{k}{r} > k$. This implies that there exists a path P_i, say P_1, such that $|N_G(x) \cap V(P_1(s,t))| > \frac{k}{r}$. Let y_1 be the leftmost neighbor of x in $P_1(s,t)$ and let y_2 be the rightmost one. Note that $|V(P_1[y_1,y_2])| \geq |N_G(x) \cap P_1(s,t)| > \frac{k}{r}$. By Claim 3, there exist k pairwise internally disjoint paths $Q_1, Q_2, \dots, Q_k$ from s to t in $\left(\bigcup_{j=1}^{k} V(P_j) - P_1(s,t)\right) \cup H_l$. Since $R' := x \cup xy_1 \cup P_1[y_1,y_2] \cup y_2x$ is a 2-connected subgraph in $G - \bigcup_{i=1}^{k} V(Q_i)$ with $|R'| > \frac{k}{r} + 1 \geq |R|$, which contradicts the choice (P1). This completes the proof of Theorem 4. $\square$

Remark: The almost identical proof of Theorem 4 implies Theorem 3.

Let us give a sketch of the proof. We follow the proof of Theorem 4.

Let $G' := G - \bigcup_{i=1}^{k} V(P_i)$, and take components $H_1, \dots, H_l$ of G' so that $(|H_1|, |H_2|, \dots, |H_l|)$ is as large as possible in lexicographic order. Suppose $l \geq 2$ and for the H_l, we do Operation 1 as in the proof of Theorem 4, and thus obtain the vertex set A and B. By the same reason as in the proof of Theorem 4, Claims 2 and 4 hold, and hence $A \cup B$ separates $\bigcup_{i=1}^{k} P_i(a_i, b_i) \cup H_l$ from H_1. However, this contradicts the connectivity condition because $|A \cup B| \leq 2k$. This proves Theorem 3. $\square$

Acknowledgments

The authors would like to thank Professor Hikoe Enomoto, who suggested the main idea of the graph showing the sharpness of Theorem 3. The authors also would like to thank Professor Katsuhiro Ota for his valuable suggestions. We are graceful to the referees for careful reading and useful comments.

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