Vertex coloring of graphs by total 2–weightings

Jonathan Hulgan[1], Jenő Lehel[12], Kenta Ozeki[3], Kiyoshi Yoshimoto[4]

[1] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152 USA
[2] Computer and Automation Research Institute, Hungarian Academy of Sciences, MTA SzTAKI Budapest, Hungary
[3] Department of Mathematics, School of Fundamental Science and Technology, Keio University, Yokohama, Japan
[4] Department of Mathematics, Nihon University, Tokyo, Japan

Abstract. An assignment of real weights to the edges and the vertices of a graph is a vertex-coloring total weighting if the total weight sums at the vertices are distinct for any two adjacent vertices. Of interest in this paper is the existence of vertex-coloring total weightings with weight set of cardinality two, a problem motivated by the conjecture that every graph has a such a weighting using the weights 1 and 2. Here we prove the existence of such weightings for certain families of graphs using any two distinct non-negative real weights.

Key words. Adjacent-vertex distinguishing Total weighting Vertex-coloring

1. Introduction

Let $G = (V, E)$ be a graph and $w : (V \cup E) \to S$, $S \subseteq \mathbb{R}$, be an assignment of real weights to the edges and vertices of G. For every $v \in V$, define the total weight sum $W(v) = w(v) + \sum_{uv \in E} w(uv)$, which we call the color of v. If $W(u) \neq W(v)$, for all $uv \in E$, then we say that w is a Vertex-Coloring Total S-Weighting, in short an S-VCTW of G. An obvious question to ask is how can S be bounded, either by restricting the number of weights in S or by minimizing the maximum absolute value of the weights in S.

This concept was introduced by Przybyło et al. in [6]. A similar concept of vertex-coloring edge weightings was introduced earlier by Karoński et al. [5] and investigated further by Addario-Berry et al. [1] and by Kalkowski et al. [4]. Additionally, this concept is related to the adjacent-vertex distinguishing total colorings (AVDTC) introduced by Zhang et al. [9], and investigated by others, including Wang [8] and Hulgan [3]. VCTW's can be considered to be an extension of AVDTC's by replacing the colors with weights and removing the restriction that the total coloring of the vertices and edges of a graph be proper.

In [6] Przybyło and Woźniak conjecture that every simple graph has a $\{1, 2\}$-VCTW. So far, the existence of a $\{1, 2\}$-VCTW has been shown for bipartite graphs, complete graphs, 3− and 4-regular graphs, and 3-colorable graphs (see [6], [7]). It has been recently shown by Kalkowski et al. that every graph has a $\{1, 2, 3\}$-VCTW, as reported in [4].

In this paper, we focus on weightings that use only two weights that we also call a 2-VCTW. It is our working hypothesis that every graph has a 2-VCTW for any two distinct non-negative real weights a and b. To that end, we examined a number of common families of graphs. Following a few comments in Sections 2 pertaining to regular graphs, in Section 3 we prove the existence of an $\{a, b\}$-VCTW for bipartite graphs (Proposition 2) and for complete multipartite graphs (Theorem 1). In Section 4 we discuss and solve the case of 3-chromatic graphs (Theorem 2). In Section 5 the existence of a $\{0, 1\}$-VCTW is shown for graphs with maximum degree at most 4. A related concept of irregular $\{0, 1\}$-weightings with a characterization result (Theorem 4) in Section 6 concludes the paper.

2. Regular graphs

In general, it is not clear that the existence of one 2-VCTW implies the existence of another. For sure, we cannot in general simply apply a bijection between the two weight sets. However, this simple method works for regular graphs. Given a graph G with total S-weighting w, for every $v \in V(G)$ define the multi-set $C(v) = \{w(v)\} \cup \{\bigcup_{uv \in E} w(uv)\}$. We say that w is a vertex-coloring total S-labeling (S-VCTL) of G if for each edge $uv \in E$, $C(u) \neq C(v)$.

Lemma 1. *Let G be a k-regular graph. Then for distinct $a, b \in \mathbb{R}$, G has an $\{a, b\}$-VCTW if and only if it has an $\{a, b\}$-VCTL.*

Proof. Let $w : (V \cup E) \to \{a, b\}$. It is clear that for $uv \in E$, if $C(u) = C(v)$, then $W(u) = W(v)$. Suppose $W(u) = W(v)$. Assume u has m elements weighted a and n elements weighted b; similarly, assume v has m' elements weighted a and n' weighted b. Observe that $m + n = m' + n' = k + 1$ and $am + bn = am' + bn'$, since $W(u) = W(v)$. It follows from these two observations that $(m - m') + (n - n') = 0$ and $(m - m')a + (n - n')b = 0$; we conclude that $c(a - b) = 0$, where $c = m - m' = n' - n$, and so $m = m'$ and $n = n'$ since a and b are distinct. Therefore $C(u) = C(v)$. □

Proposition 1. *Let G be a k-regular graph. Then for $a, a', b, b' \in \mathbb{R}$, where $a \neq b$ and $a' \neq b'$, G has an $\{a, b\}$-VCTW if and only if it has an $\{a', b'\}$-VCTW.*

Proof. By Lemma 1, if G has an $\{a, b\}$-VCTW then it has an $\{a, b\}$-VCTL. We can obtain an $\{a', b'\}$-VCTL simply by replacing each a with a' and b with b'. Applying the lemma again gives us an $\{a', b'\}$-VCTW. By symmetry, the reverse implication is also true. □

3. Ad-hoc weightings

A number of well-known families of graphs have relatively simple and instructive proofs of the existence of 2-VCTW's. In most instances, we require that a and b be distinct non-negative integers. As we note in the next section, this is sufficient to prove the existence of 2-VCTW's with non-negative real numbers. However, note that in the proof for complete multipartite graphs, we allow a and b to be any two distinct real numbers.

Proposition 2. *Let G be a bipartite graph and a, b be distinct non-negative integers. Then G has an $\{a, b\}$-VCTW.*

Proof. Let V_1, V_2 denote the partite sets of G and assume $a < b$. If $a = 0$, weight the vertices of V_1 with b and all other elements with 0; since for $v_1 \in V_1$ and $v_2 \in V_2$ we have $W(v_1) = b$ and $W(v_2) = 0$, this results in a proper coloring. Assume a and b are nonzero. Weight the vertices of V_1 with a and all other elements with b. Observe that for $v_1 \in V_1$, $W(v_1) \equiv a \mod b$ and for $v_2 \in V_2$, $W(v_2) \equiv 0 \mod b$. Thus for $uv \in E$, $W(u) \neq W(v)$. □

Corollary 1. *Trees have an $\{a, b\}$-VCTW for any two distinct non-negative integer weights a and b.*

Corollary 2. *Every graph with maximum degree 2 has an $\{a, b\}$-VCTW for any two distinct non-negative integer weights a and b.*

Proof. By Proposition 2, the result is true for paths and even cycles. Suppose G is an odd cycle $(v_1, v_2, \ldots, v_{2k+1})$. Let $w(v_1) = w(v_1v_2) = w(v_{2j}) = a$ for $2 \leq j \leq k$, and weight all other elements with b. This results in a weighting of G satisfying $W(v_1) = 2a + b$, $W(v_{2j}) = a + 2b$ and $W(v_{2j+1}) = 3b$, for $1 \leq j \leq k$. Since a and b are distinct, this gives a proper vertex coloring of G. □

Theorem 1. *Let G be a complete multipartite graph and let $a < b$ be real numbers. Then G has an $\{a, b\}$-VCTW.*

Proof. Let $V_1, V_2, \ldots, V_\ell$ denote the independent sets of G. Let G_i be the subgraph of G induced by $\cup_{m=1}^{i} V_m$, furthermore set $k_i = |V_i|$ and $n_i = |G_i|$. We may assume that $k_2 \leq k_1$, furthermore $k_{i-2} \leq k_i$ for i odd, and $k_{i-2} \geq k_i$ for i even, $i = 3, \ldots, \ell$. We incrementally weight the elements of G as follows: at the ith step, weight V_i and its incident edges in G_i with a or b for i odd or even, respectively. Let $W_i(v)$ denote the weight sum of v at step i.

Here let v_i denote a vertex of V_i. Observe that $W_2(v_1) < W_2(v_2)$ since $a + bk_2 < b + bk_1$. For $i \geq 3$ odd, we note that $W_i(v_i) < W_i(v_{i-2})$:

$$\begin{aligned} ak_{i-1} + ak_{i-2} &< bk_{i-1} + ak_i \\ ak_{i-1} + ak_{i-2} + an_{i-3} + a &< bk_{i-1} + ak_i + an_{i-3} + a \\ a(n_{i-1} + 1) &< a(n_{i-3} + 1) + bk_{i-1} + ak_i \\ W_i(v_i) &< W_{i-2}(v_{i-2}) + bk_{i-1} + ak_i \\ W_i(v_i) &< W_i(v_{i-2}). \end{aligned}$$

Similarly, for $i \geq 4$ even, $W_i(v_i) > W_i(v_{i-2})$:

$$\begin{aligned} bk_{i-1} + bk_{i-2} &> ak_{i-1} + bk_i \\ bk_{i-1} + bk_{i-2} + bn_{i-3} + b &> ak_{i-1} + bk_i + bn_{i-3} + b \\ b(n_{i-1} + 1) &> b(n_{i-3} + 1) + ak_{i-1} + bk_i \\ W_i(v_i) &> W_{i-2}(v_{i-2}) + ak_{i-1} + bk_i \\ W_i(v_i) &> W_i(v_{i-2}). \end{aligned}$$

At each step i, the weight sum of each $v \in V(G_{i-1})$ increases by the same amount; with our previous observations this implies for j odd and m even,

$$W_i(v_j) < W_i(v_{j-2}) \leq W_i(v_1) < W_i(v_2) \leq W_i(v_{m-2}) < W_i(v_m)$$

where $3 \leq j, m \leq i$. Therefore G_i has an $\{a, b\}$-VCTW for $1 \leq i \leq \ell$; since $G_\ell = G$, we have the desired result. □

Theorem 1 proves the existence of $\{a, b\}$-VCTW's for many classic families of graphs, including complete graphs, complete graphs with a matching removed, and Turán graphs.

4. $\{a,b\}$-VCTW of 3-colorable graphs

Here we assume that a and b are any non-negative integers. The following observation allows us to assume that a and b are relatively prime when seeking to find an $\{a,b\}$-VCTW of a graph.

Lemma 2. *Let G be a graph and $a, b, c \in \mathbb{R}$, $c \neq 0$. Then if G has an $\{a,b\}$-VCTW, it has an $\{ac, bc\}$-VCTW.*

Proof. We claim an $\{ac, bc\}$-VCTW can be obtained from an $\{a,b\}$-VCTW by replacing a weights with ac weights and b weights with bc weights. Let $W(v)$ denote the weight sum of v from the $\{a,b\}$-VCTW and $W'(v)$ denote the weight sum of v from the $\{ac, bc\}$-TW obtained in the manner described. Suppose this process results in two adjacent vertices u and v with $W'(u) = W'(v)$. Observe that $W'(u) = cW(u)$ and $W'(v) = cW(v)$; this implies $W(u) = W(v)$, a contradiction. □

From this fact, we may also conclude that proving the existence of a 2-VCTW for any pair of distinct non-negative integers implies the existence of a 2-VCTW for any pair of distinct non-negative real numbers. Suppose $\frac{a}{b} \neq 1$ is positive and rational; then there exists some real number c such that ac and bc are distinct non-negative integers. Thus since there exists an $\{ac, bc\}$-VCTW, there exists an $\{a,b\}$-VCTW. Suppose $\frac{a}{b} = \gamma$ is irrational; considering some $\{0,1\}$-VCTW, we may obtain a $\{\gamma, 1\}$-VCTW by replacing every 0 weight with γ. Multiplying all weights by b gives us an $\{a,b\}$-VCTW.

Lemma 2 proves to be quite useful when handling larger weight values in Theorem 2, in particular when $b \geq 4$. For $a, b \in \{1,2,3\}$, the conclusion is true, but requires a more intricate approach. In fact, here we actually prove that there exists an $\{a,b\}$-weighting such that adjacent vertices have distinct weight sums modulo 3. The proof of this lemma is given at the end of the section.

Lemma 3. *Let G be a 3-colorable graph and $a, b \in \{1,2,3\}$ be distinct. Then G has an $\{a,b\}$-weighting w such that $W(u) \not\equiv W(v) \mod 3$ for every $uv \in E$.*

Theorem 2. *Let G be a 3-colorable graph and $a < b$ be non-negative integers. Then G has an $\{a,b\}$-VCTW.*

Proof. By Lemma 2 we may assume $(a,b) = 1$. The $\{0,1\}$ case follows from Lemma 3 by finding such a $\{1,3\}$-VCTW and taking all weights modulo 3. The case $a, b \in \{1,2,3\}$ follows directly from Lemma 3. Suppose $b \geq 4$. Let S_1, S_2, and S_3 denote the color classes for a greedy 3-coloring of G (that is, every vertex in S_i has a neighbor in S_j where $j < i$). In S_3, weight all vertices and incident edges with b. In S_2, weight each vertex a and exactly one edge connecting the vertex to a neighbor in S_1 with a; weight all other incident edges with b. In S_1, all incident edges should have been previously weighted. Weight each vertex of S_1 so that $W(v) \not\equiv 0$ or $2a$ modulo b: if the weight sum of the incident edges is congruent to 0 or $2a \mod b$, then weight the vertex a, otherwise weight it b. By our assumption that $b \geq 4$ and $(a,b) = 1$, it follows that $0, a, 2a, 3a \mod b$ are all distinct values. Thus $W(v_i) \neq W(v_j)$ for $v_i \in S_i$ and $v_j \in S_j$ where $1 \leq i < j \leq 3$. □

Proof. (of Lemma 3) Let $x \in V(G)$ be a non-cut vertex of G and f be a 3-coloring of G. Let S_1, S_2, and S_3 denote the color classes of f where $x \in S_1$; let v_i denote an arbitrary vertex contained in S_i. We claim there exists an $\{a,b\}$-TW of G where $W(v_i) \equiv i$

mod 3 for every $v_i \in V(G) \setminus \{x\}$. Suppose not; let w be an $\{a,b\}$-TW with the fewest number of vertices such that $W(v_i) \not\equiv i \mod 3$. Observe if $W(v_i) \not\equiv i \mod 3$, then either $W(v_i) \equiv i + a - b \mod 3$ or $W(v_i) \equiv i + b - a \mod 3$; furthermore, observe that in the first case $w(v_i) = b$ and in the second $w(v_i) = a$, otherwise a correct weighting could be obtained simply by changing the weight of v_i. Since these two cases are symmetric, we may focus only on the first.

Consider the following scheme for reducing the number of incorrectly weighted vertices or moving an incorrect weighting from v_i to a neighbor v_j. If $w(v_iv_j) = a$, change its weight to b; if $w(v_iv_j) = b$, change its weight to a and change $w(v_i)$ to a. Observe in both of these cases, this change results in a correct weighting for v_i and only affects the weighting of v_j. By using this process, we may "push" an incorrect weighting of u along a uv-path by first moving the incorrect weighting of u to its neighbor in the path and then iterating this process, starting at the next incorrectly weighted vertex on the path.

When this process completes, there can be at most one incorrectly weighted vertex in this path, namely v. If w has more than one incorrectly weighted vertex, this number can be reduced by one by pushing a bad weighting along a path between two of these vertices. Since w contains the fewest number of such vertices, we conclude it can have at most one incorrectly weighted vertex, and furthermore we may push it to any vertex in the graph; by pushing the bad vertex to x, we prove the claim.

If $W(x) \equiv 1 \mod 3$, we are done. Suppose not; without loss of generality, let $W(x) \equiv 1+a-b \mod 3$. If $w(x) = a$, then a correct weighting could be obtained by changing $w(x)$ to b; assume $w(x) = b$. Suppose x has two edges xy and xz with the same weight. Since x is not a cut vertex, there exists a cycle C containing x, y, and z. If we push the incorrect weighting of x along C so that at some point no incorrect weighting is induced, we are done; otherwise, suppose we push the weighting back to x, that is, all the way around C. Call these new weights w'. If $w(xy) = w(xz) = a$, then $w'(xy) = w'(xz) = b$; by letting $w'(x) = a$, we have $W'(x) = W(x) + 2(b-a) + (a-b) \equiv 1 \mod 3$. If $w(xy) = w(xz) = b$, then $w'(xy) = w'(xz) = a$; by keeping $w'(x) = b$, we have $W'(x) = W(x) + 2(a-b) \equiv 1 \mod 3$.

If no $xy, xz \in E(G)$ exist with the same weight, then either $d(x) = 1$ or $d(x) = 2$ and x has exactly one edge of each weight. If $d(x) = 1$, it has a unique neighbor y; by choosing $w(x)$ appropriately, we may conclude $W(x) \not\equiv W(y)$, thus proving the result. If $d(x) = 2$, then $W(x) = a + 2b \equiv 1 + a - b \mod 3$; but this implies $3b \equiv 1 \mod 3$, a contradiction. □

5. $\{0,1\}$-VCTW of graphs with maximum degree 4

Our proof of the existence of a $\{0,1\}$-VCTW for graphs with maximum degree 4 follows easily from a fortuitous fact concerning the existence of a special $\{0,1\}$-VCTW for graphs with maximum degree 3, which we will prove momentarily. For the purposes of $\{0,1\}$ weightings, we assume that weighted elements are assigned weight 1; if elements are not explicitly assigned a weight, assume that they are weighted 0. Let $\Delta(G)$ denote the maximum degree of graph G.

Lemma 4. *If G is a graph with $\Delta(G) \le 3$ then it has a $\{0,1\}$-VCTW such that $W(v) \ge 1$ for every $v \in V(G)$.*

Theorem 3. *If G is a graph with $\Delta(G) \le 4$ then it has a $\{0,1\}$-VCTW.*

Proof. Let G be a graph with $\Delta(G) \leq 4$. We take a maximum independent set I of G and let H be the subgraph of G induced by $V(G) \setminus I$. Since every vertex in $V(H)$ has a neighbor in I, we have $\Delta(H) \leq 3$. By Lemma 4, H has a $\{0,1\}$-VCTW such that no vertex is colored by 0. Extend this to a weighting of G by weighting every other element with 0. Every vertex in I has color 0, and every vertex in H has strictly positive color distinct from that of its neighbors in H. Thus adjacent vertices have distinct colors. □

Proof. (of Lemma 4) Consider a greedy 3-coloring of G into color classes S_1, S_2, and S_3. We will consider two types of vertices in S_3: define T_1 to be the set of vertices in S_3 which have exactly one neighbor in S_1, and define T_2 to be the set of vertices in S_3 with two neighbors in S_1. Additionally, define Y to be the set of vertices in S_2 with two neighbors in S_3, at least one of which is in T_2. We weight G so that vertices in S_1 and T_1 have odd color while those in S_2 and T_2 have even color; furthermore, we distinguish adjacent vertices with the same color parity. We incrementally weight the elements of G in three steps. At each step, let $L(v)$ denote the sum of all incident edge weights up to that step.
Step 1. For each $v \in S_2 \setminus Y$ weight exactly one edge incident with a vertex in S_1 by 1. Weight all edges between S_1 and T_2 by 1.
Step 2. Let $v \in S_1$ have exactly one neighbor $x \in T_1$. Note that $L(v) \leq 2$ since v can have at most two neighbors in $(S_2 \setminus Y) \cup T_2$. If $L(v) = 0$, let $u \in S_2$ be a neighbor of x and weight vx, ux, and x by 1. If $L(v) = 1$, weight v and vx by 1; if $L(v) = 2$, weight vx by 1.

Now let $v \in S_1$ have exactly two neighbors $x_1, x_2 \in T_1$. Note that $L(v) \leq 1$ since v can have at most one neighbor in $(S_2 \setminus Y) \cup T_2$. If $L(v) = 0$, weight vx_1, vx_2, and v by 1. If $L(v) = 1$, weight vx_1 and vx_2 by 1. If v has three neighbors $x_1, x_2, x_3 \in T_1$ then $L(v) = 0$ and we weight vx_1, vx_2, and vx_3 by 1. If $v \in S_1$ has no neighbors in T_1, choose $w(v)$ so that $W(v)$ is odd; that is, if $L(v) = 0$ or 2, let $w(v) = 1$, otherwise let $w(v) = 0$.
Step 3. Let $v \in S_2 \setminus Y$. If v has exactly one neighbor $x \in T_2$, then $L(v) = 1$, otherwise a higher value would imply v has a neighbor in T_1, and thus $v \in Y$; weight vx and x by 1. If v has no such neighbor in T_2, choose $w(v)$ so that $W(v)$ is even; that is, if $L(v) = 1$ or 3, let $w(v) = 1$, otherwise let $w(v) = 0$. Note that $L(v) \geq 1$ from Step 1, so $W(v) > 0$. Suppose $v \in Y$ has exactly one neighbor $x \in T_2$; note that $0 \leq L(v) \leq 1$. If $L(v) = 0$, weight v, x, and vx by 1; if $L(v) = 1$, weight x and vx by 1. Suppose v has two neighbors $x_1, x_2 \in T_2$. In this case $L(v) = 0$ and we weight vx_1, vx_2, x_1, and x_2 by 1.

In Step 3 we have weighted only elements in or incident to S_2 and T_2, so vertices distinguished in Step 2 remain distinguished; thus we obtain a $\{0,1\}$-VCTW of G. □

Corollary 3. *Let G be a 4-regular graph and a, b be distinct real numbers. Then G has an $\{a,b\}$-VCTW.*

Proof. By Theorem 3, G has a $\{0,1\}$-VCTW. By Proposition 1, this implies the existence of an $\{a,b\}$-VCTW. □

6. Vertex distinguishing total weightings

Almost every previous irregular coloring or weighting problem has come in two forms: distinguish adjacent vertices or distinguish all vertices. So far we have investigated in this paper the former such problem of S-VCTW. The latter problem has also been investigated, having been introduced by Bača et al. in [2]. We call an assignment $w : (V \cup E) \to S$ a Vertex Distinguishing Total S-Weighting, S-VDTW for short, if $W(u) \neq W(v)$, for every $u, v \in V$. Here we classify graphs that have a $\{0,1\}$-VDTW.

Let H_n be the graph defined as follows: $V(H_n) = \{v_1, v_2, \ldots, v_n\}$, and $E(H_n) = \{v_j v_k : j + k \geq n + 1\}$.

Proposition 3. *For $n \geq 3$, $H_n - v_n \cong H_{n-2} \cup \{v_0\}$.*

Proof. Denote $u_j = v_{j-1}$, $j = 1, \ldots, n-1$, a vertex of $H_{n-2} \cup \{v_0\}$, and use v_i, $i = 1, \ldots, n-1$, to denote the vertices of $H_n - v_n$. We claim that the bijection $v_i \leftrightarrow u_i, 1 \leq i \leq n-1$, is an isomorphism between $H_n - v_n$ and $H_{n-2} \cup \{v_0\}$.

By the definition of H_n, we have $v_i v_j \in E(H_n - v_n)$ if and only if $i+j \geq n+1$. Similarly, by the definition of H_{n-2}, $u_k u_\ell \in E(H_{n-2} \cup \{v_0\})$ if and only if $v_{k-1} v_{\ell-1} \in E(H_{n-2})$ if and only if $k + \ell \geq n+1$. Therefore $v_i v_j \in E(H_n - v_n)$ if and only if $u_i u_j \in E(H_{n-2} \cup \{v_0\})$. □

The graphs H_n have several similar properties that can be obtained easily by the definition. There is exactly one maximum matching in H_n, the set $M_n = \{v_j v_k : j + k = n + 1\}$. Furthermore, $\overline{H_n} \cong H_n \setminus M_n \cong H_{n-1} \cup \{v_0\}$.

The graph H_n is almost degree irregular, it has all degrees $1, \ldots, n-1$, and just two vertices are of the same degree $\lfloor \frac{n}{2} \rfloor$. It is easy to see that defining $w(v_i v_j) = 1$, for $1 \leq i < j \leq n$, $w(v_i) = 0$ for $i \leq \lfloor \frac{n}{2} \rfloor$, and $w(v_i) = 1$ for $i > \lfloor \frac{n}{2} \rfloor$ we obtain a $\{0,1\}$-VDTW of H_n. Note that the construction of H_n and its subsequent weighting can be considered as the standard $\{0,1\}$-VCTW of the clique K_n that is obtained as a corollary of Theorem 1.

Proposition 4. *Let G be a graph of order n. If G contains a copy of H_n or H_{n-1}, then it has a $\{0,1\}$-VDTW.*

Proof. Weight all elements of the copy of H_n or H_{n-1} according to the previously described weighting; weight all other elements of G with 0. If G contains a copy of H_n, the copy of H_n spans G; since all extra edges are weighted 0, every vertex has a unique color by the weighting of H_n. If G contains a copy of H_{n-1}, let v_0 be the vertex not saturated by H_{n-1}. Clearly, $W(v_0) = 0$ and the vertices in the copy of H_{n-1} will have distinct positive weights, thus we obtained a $\{0,1\}$-VDTW. □

Proposition 5. *Let G be a graph of order n and let w be a $\{0,1\}$-VDTW. If $W(v) = 0$ for some $v \in V(G)$, then G contains a copy of H_{n-1}, and if $W(v) = n$ for some $v \in V(G)$, then G contains a copy of H_n.*

Proof. We proceed by induction on n. The statement is clearly true for $n = 2$, since $H_2 \cong P_2$ and H_1 is a single vertex. Assume the statement is true for $n = k-1$. Let G be a graph of order k and let w be a $\{0,1\}$-VDTW of G. Observe that by the pigeonhole principle, G must contain either a vertex v with $W(v) = 0$ or $W(v) = k$.

Suppose G has a vertex v such that $W(v) = 0$. Let $G' = G - v$. Observe that w', the restriction of the weighting w of G to G', is a $\{0,1\}$-VDTW of G'; otherwise w would not be a $\{0,1\}$-VDTW of G since all edges incident to v in G are weighted 0. Furthermore, there cannot exist a vertex in G' with color 0, otherwise G would have two vertices with color 0; so there must exist a vertex v' in G' such that $W'(v') = W(v') = k-1$. Since $|V(G')| = k-1$, by hypothesis G' contains a copy of H_{k-1}, and thus so does G.

Suppose G has a vertex v such that $W(v) = k$. Let $G' = G - v$. Observe that w', the restriction of the weighting w of G to G', is a $\{0,1\}$-VDTW of G'; otherwise w would not be a $\{0,1\}$-VDTW of G since v must be adjacent to all other vertices in G and each incident edge is weighted 1. Furthermore, there cannot exist a vertex in G' with color

$k-1$, otherwise G would have two vertices with color k; therefore there exists a vertex v' in G' such that $W'(v') = W(v) - 1 = 0$. Since $|V(G')| = k-1$, by hypothesis G' contains a copy of H_{k-2} that misses some vertex. By Proposition 3, since v is adjacent to every vertex in G', G must contain a copy of H_k. □

By Propositions 4 and 5, we conclude the following:

Theorem 4. *A graph G of order n has a $\{0,1\}$-VDTW if and only if G contains a copy of H_n or H_{n-1}.*

7. Closing Comments

Here we have focused on 2-VCTW's of graphs using non-negative real values. In fact, our results from Section 4 imply the existence of 2-VCTW's in 3-colorable graphs for a slightly larger set of numbers, namely all real multiples of pairs of integers not congruent modulo 3. For example, considering $\{-1,2\}$-VCTW's appears to be a tricky case. This motivates the following problem.

Question 1. Do there exist distinct real numbers a and b and a graph G such that G has no $\{a,b\}$-VCTW?

In [4], the authors provide an elegant proof of the existence of a $\{\alpha-\beta, \alpha, \alpha+\beta\}$-VCTW for every graph G and every distinct real α and β where $\beta \neq 0$.

Question 2. For every graph G and distinct real numbers α, β, and γ, does G have an $\{\alpha,\beta,\gamma\}$-VCTW?

References

1. Louigi Addario-Berry, Ketan Dalal, Colin McDiarmid, Bruce Reed, and Andrew Thomason. Vertex-colouring edge-weightings. *Combinatorica*, 27(1):1–12, 2007.
2. Martin Bača, Stanislaf Jendrol, Mirka Miller, and Joseph Ryan. On irregular total labelings. *Discrete Mathematics*, 307:1378–1388, 2007.
3. Jonathan Hulgan. Concise proofs for adjacent vertex-distinguishing total colorings. *Discrete Mathematics*, In Press, Corrected Proof, 2008.
4. Maciej Kalkowski, MichałKaronński, and Florian Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. Preprint, 2009.
5. Michał Karoński, Tomasz Łuczak, and Andrew Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91:151–157, 2004.
6. Jakub Przybyło and Mariusz Woźniak. 1,2 conjecture. Preprint MD 024, 2007.
7. Jakub Przybyło and Mariusz Woźniak. 1,2 conjecture, ii. Preprint MD 026, 2007.
8. Haiying Wang. On the adjacent vertex-distinguishing total chromatic numbers of the graphs with $\Delta(G) = 3$. *Journal of Combinatorial Optimization*, 14:87–109, 2007.
9. Zhongfu Zhang, Xiang'en Chen, Jingwen Li, Bing Yao, Xinzhong Lu, and Jianfang Wang. On adjacent-vertex-distinguishing total coloring of graphs. *Science in China Series A Mathematics*, 48:289–299, 2005.