

Spanning trees with a bounded number of leaves in a claw-free graph

Mikio Kano¹, Aung Kyaw^{2*}, Haruhide Matsuda³, Kenta Ozeki⁴,
Akira Saito⁵ and Tomoki Yamashita⁶

¹ Department of Computer and Information Sciences
Ibaraki University, Hitachi, Ibaraki, 316-8511, Japan

² Department of Mathematics
East Yangon University, Yangon, Myanmar

³ Department of Mathematics,
Shibaura Institute of Technology,
Fukasaku, Minuma-ku, Saitama 337-8570, Japan

⁴ National Institute of Informatics,
Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

⁵ Department of Computer Science and System Analysis
Nihon University,
Sakurajosui, Setagaya-Ku, Tokyo, 156-8550, Japan

⁶ College of Liberal Arts and Sciences,
Kitasato University,
Kitasato, Minami-ku, Sagamihara 252-0373, Japan

*This paper was written while the second author is on a Matsumae International Foundation research fellowship at the Department of Computer and Information Sciences, Ibaraki University. Hospitality and financial support are gratefully acknowledged.

Abstract

For a graph H and an integer $k \geq 2$, let $\sigma_k(H)$ denote the minimum degree sum of k independent vertices of H . We prove that if a connected claw-free graph G satisfies $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning tree with at most k leaves. We also show that the bound $|G| - k$ is sharp and discuss the maximum degree of the required spanning trees.

Keywords: spanning tree, leaf, degree sum, claw-free graphs

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, we consider only simple graphs, which have neither loops nor multiple edges. We write $|G|$ for the order of G , that is, $|G| = |V(G)|$. For a vertex v of G , we denote by $\deg_G(v)$ the degree of v in G . A vertex of degree one is called an *end-vertex*, and an end-vertex of a tree is usually called a *leaf*. A vertex set S of G is called *independent* if no two vertices of S are adjacent in G . The minimum degree sum of k independent vertices of G is denoted by $\sigma_k(G)$, that is, if G has k independent vertices, let

$$\sigma_k(G) = \min_S \left\{ \sum_{x \in S} \deg_G(x) : S \text{ is an independent set of } G \text{ with } k \text{ vertices} \right\}.$$

If G does not have k independent vertices, we define $\sigma_k(G) = +\infty$. A graph G is said to be *claw-free* if it contains no $K_{1,3}$ as an induced subgraph. The connectivity, the independence number and the minimum degree of G are denoted by $\kappa(G)$, $\alpha(G)$ and $\delta(G)$, respectively.

By Dirac's Theorem, every graph G of order at least three with $\delta(G) \geq \frac{1}{2}|G|$ has a hamiltonian cycle. As an immediate corollary, we can prove that every graph G with $\delta(G) \geq \frac{1}{2}(|G| - 1)$ has a hamiltonian path. For general graphs, the bound $\frac{1}{2}(|G| - 1)$ is sharp. For example, for a positive integer m , the complete bipartite graph with partite sets of order m and $m + 2$ satisfies $\delta(G) = m = \frac{1}{2}(|G| - 2)$, but G has no hamiltonian path. However, Matthews and Sumner [5] proved that if we restrict ourselves to

the class of claw-free graphs, a considerably smaller bound on minimum degree guarantees the existence of a hamiltonian path.

Theorem 1 (Matthews and Sumner [5]) *Let G be a connected claw-free graph. If $\delta(G) \geq (|G| - 2)/3$, then G has a hamiltonian path.*

Ore's Theorem states that every graph of order at least three with $\sigma_2(G) \geq |G|$ has a hamiltonian cycle. It extends Dirac's Theorem, and implies as a corollary that every graph G with $\sigma_2(G) \geq |G| - 1$ has a hamiltonian path.

A path of order at least two can be interpreted as a tree having exactly two leaves. From this point of view, a hamiltonian path of a graph of order at least two is a spanning tree with exactly two leaves. This interpretation may lead us to consider a spanning tree with a bounded number of leaves. Actually, Broersma and Tuinstra [1] gave a sufficient condition for a connected graph to have such a spanning tree.

Theorem 2 (Broersma and Tuinstra [1]) *Let $k \geq 2$ be an integer and let G be a connected graph of order at least two. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning tree with at most k leaves.*

The previous corollary of Ore's Theorem corresponds to the case $k = 2$ of the above theorem.

Broersma and Tuinstra also proved that the bound $|G| - k + 1$ of $\sigma_2(G)$ is sharp. However, in view of Theorem 1, for claw-free graphs, a much weaker condition may yield the same conclusion as in Theorem 2. Motivated by this observation, we study a degree sum condition for a claw-free graph to have a spanning tree with a bounded number of leaves, and give the following theorem.

Theorem 3 *Let $k \geq 2$ be an integer and let G be a connected claw-free graph. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning tree with at most k leaves.*

Note that Theorem 1 is a corollary of the case $k = 2$ of the above theorem.

In the next section, we prove the above theorem. In Section 3, we investigate the maximum degree of a spanning tree and prove that under the same assumption as in Theorem 3, G has a spanning tree of maximum degree at most three with at most k leaves. In Section 4, we give concluding remarks.

Before proving Theorem 3, we first show that the bound $|G| - k$ of $\sigma_{k+1}(G)$ is sharp. Consider a graph G constructed from one complete graph K_{k+1} and $k + 1$ complete graphs K_m , $m \geq 2$, by identifying one vertex of each K_m with one distinct vertex of K_{k+1} (see Figure 1). Then G is claw-free and satisfies $\sigma_{k+1}(G) = |G| - k - 1$, but G has no spanning tree with at most k leaves.

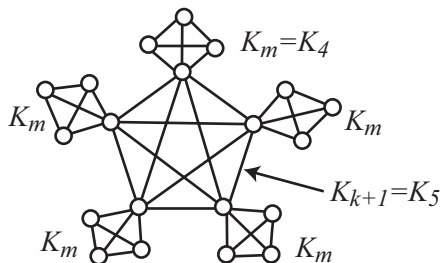


Figure 1: A connected claw-free graph G that has no spanning tree with at most k leaves and satisfies $\sigma_{k+1}(G) = |G| - k - 1$.

Some other results on spanning trees having at most k leaves can be found in [2] and [8].

2 Proof of Theorem 3

We begin with some additional notation. For a vertex v of a graph G , the neighborhood of v in G is denoted by $N_G(v)$. For a vertex set X of G , we write $N_G(X) = \bigcup_{x \in X} N_G(x)$, and the subgraph of G induced by X is denoted by $\langle X \rangle_G$. We write $G - X$ for $\langle V(G) - X \rangle_G$, and for a vertex v , $G - \{v\}$ is briefly denoted by $G - v$.

The complete graph of order n is denoted by K_n . The complete bipartite

graph with bipartition (X, Y) , where $|X| = m$ and $|Y| = n$, is denote by $K_{m,n}$ or $K(m, n)$. The graph constructed from two complete graphs K_m and K_n by identifying one vertex of K_m with one vertex of K_n is called a *double complete graph* and denoted by $DC(m, n)$, where $m, n \geq 2$. The common vertex of K_m and K_n is called the *center*, and the other vertices are called *non-central vertices* (See Figure 2). Note that the order of $DC(m, n)$ is $m+n-1$, and the path of order three is a double complete graph $DC(2, 2)$. Let \mathcal{D} denote the set of all double complete graphs.

When we consider a path or a cycle, we always assign an orientation. Let W be a path or a cycle, and let $v \in V(W)$. Then we denote by $v^{-(W)}$ and $v^{+(W)}$ the predecessor and the successor of W , respectively. We write $v^{--(W)}$ instead of $(v^{-(W)})^{-(W)}$. For $A \subset V(W)$, let $A^{-(W)} = \{v^{-(W)} : v \in A\}$. If W is clear from the context, we often omit “ (W) ” and write v^- , v^+ , v^{--} and A^- instead of $v^{-(W)}$, $v^{+(W)}$, $v^{--(W)}$ and $A^{-(W)}$, respectively. A path which starts at a vertex u and ends at a vertex v is called a uv -path. For a path P and vertices $u, v \in V(P)$, a subpath of P with ends u and v is denoted by $P(u, v)$. For subgraphs H_1 and H_2 of a graph G , we define H_1+H_2 by $H_1+H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. When we consider this operation, an edge is often considered as a subgraph isomorphic to K_2 . For example, for $uv \in E(G)$, $H_1 + uv = (V(H_1) \cup \{u, v\}, E(H_1) \cup \{uv\})$. For further explanation of terminologies and notation, we refer the reader to [9].

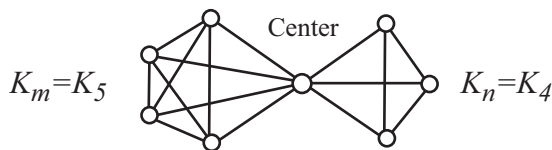


Figure 2: The double complete graph $DC(m, n)$, whose order is $m + n - 1$.

Enomoto [3], Jung [4] and Nara [6] implicitly characterized the connected graphs G such that G satisfies $\deg_G(x) + \deg_G(y) \geq |G| - 1$ for every pair of vertices x and y of G which are end-vertices of some hamiltonian

path of G , but G has no hamiltonian cycle. The next lemma is a corollary of this characterization. We give its proof for the self-containedness of the paper.

Lemma 4 *Let G be a claw-free graph having a hamiltonian path. Suppose that $\deg_G(x) + \deg_G(y) \geq |G| - 1$ for every pair of vertices x and y which are end-vertices of some hamiltonian path. Then G has a hamiltonian cycle, or G is a double complete graph.*

Proof. Assume G has no hamiltonian cycle. Let P be a hamiltonian path and let x and y be the first and the last vertices of P , respectively. By the assumption, $xy \notin E(G)$. If $N_G(x)^- \cap N_G(y) \neq \emptyset$, then $P(x, v) + vy + P(y, v^+) + v^+x$, where $v \in N_G(x)^- \cap N_G(y)$, is a hamiltonian cycle, a contradiction. Thus, $N_G(x)^- \cap N_G(y) = \emptyset$. Since $N_G(x)^- \cup N_G(y) \subset V(G) - \{y\}$ and $|N_G(x)^- \cup N_G(y)| = |N_G(x)^-| + |N_G(y)| = |N_G(x)| + |N_G(y)| = \deg_G(x) + \deg_G(y) \geq |G| - 1$, we have $N_G(x)^- \cup N_G(y) = V(G) - \{y\}$ and $\deg_G(x) + \deg_G(y) = |G| - 1$. On the other hand, since $N_G(x) \cup N_G(y) \subset V(G) - \{x, y\}$ and $\deg_G(x) + \deg_G(y) \geq |G| - 1$, we have $N_G(x) \cap N_G(y) \neq \emptyset$. We consider two cases.

Case 1. $|N_G(x) \cap N_G(y)| = 1$.

In this case, $N_G(x) \cup N_G(y) = V(G) - \{x, y\}$. Let $N_G(x) \cap N_G(y) = \{z\}$. Since $N_G(x)^- \cap N_G(y) = \emptyset$ and $N_G(x) \cup N_G(y) = V(G) - \{x, y\}$, $v \in N_G(x) - \{x^+\}$ implies $v^- \in N_G(x)$. This implies $P(x^+, z) \subset N_G(x)$. Similarly, $P(z, y^-) \subset N_G(y)$. Since $N_G(x) \cap N_G(y) = \{z\}$, we have $N_G(x) = P(x^+, z)$ and $N_G(y) = P(z, y^-)$.

Let $x_1 \in P(x^+, z^-)$. Then $x_1^+ \in N_G(x)$ and $P(x_1, x) + xx_1^+ + P(x_1^+, y)$ is a hamiltonian path of G . If $N_G(x_1) \cap P(z^+, y) \neq \emptyset$, then $P(x, x_1) + x_1y_1 + P(y_1, y) + yy_1^- + P(y_1^-, x_1^+) + x_1^+x$, where $y_1 \in N_G(x_1) \cap P(z^+, y)$, is a hamiltonian cycle of G , a contradiction. Therefore, $N_G(x_1) \subset P(x, z) - \{x_1\}$. Since $\deg_G(x_1) + \deg_G(y) \geq |G| - 1$ by the assumption, we have $N_G(x_1) \cap N_G(y) = \{z\}$. The we can apply the same argument as in the previous paragraph to x_1 and y , and obtain $N_G(x_1) = P(x, z) - \{x_1\}$. This implies that z is a cutvertex of G and $P(x, z)$ induces a complete graph.

By symmetry, $P(z, y)$ also induces a complete graph. Therefore, G is a double complete graph.

Case 2. $|N_G(x) \cap N_G(y)| \geq 2$.

In this case, there exist $x_0 \in N_G(x)$ and $y_0 \in N_G(y)$ such that $x_0 \in P(y_0^+, y)$. Choose such x_0 and y_0 so that $P(y_0, x_0)$ is as short as possible. Since $N_G(x)^- \cap N_G(y) = \emptyset$, $y_0^+ \neq x_0$.

Since $xy \notin E(G)$ and $N_G(x)^- \cup N_G(y) = V(G) - \{y\}$, x_0^{--} exists and $x_0^{--} \in N_G(x)^- \cup N_G(y)$. Since $x_0^- \notin N_G(x)$ by the choice of (x_0, y_0) , $x_0^{--} \in N_G(y)$. Again by the choice of (x_0, y_0) , we have $y_0 = x_0^{--}$. Since $P(y_0^+, x) + xx_0 + P(x_0, y)$ and $P(y_0^+, y) + yy_0 + P(y_0, x)$ are both hamiltonian paths, we can apply the same argument as that for P to these paths, and obtain $\deg_G(y_0^+) + \deg_G(y) = \deg_G(y_0^+) + \deg_G(x) = \deg_G(y) + \deg_G(x) = |G| - 1$, which yields $\deg_G(x) = \deg_G(y) = \deg_G(y_0^+) = \frac{1}{2}(|G| - 1)$.

Let $C = P(x, y_0) + y_0y + P(y, x_0) + x_0x$. Then $V(C) = V(G) - \{y_0^+\}$. Let $C = v_0v_1 \dots v_{|G|-2}v_0$. If y_0^+ is adjacent to a consecutive vertices of C , then we can insert y_0^+ to this cycle to obtain a hamiltonian cycle of G , contradicting the assumption. Since $\deg_G(y_0^+) = \frac{1}{2}(|G| - 1)$, y_0^+ is adjacent to every other vertex of C . Let $v_i \in N_G(y_0^+)$. Then $v_{i-2} \in N_G(y_0^+)$. Since $\{v_{i-1}, v_{i+1}, y_0^+\} \subset N_G(v_i)$ and G is claw-free, we have $v_{i-1}v_{i+1} \in E(G)$. Then by replacing $v_{i-2}v_{i-1}v_iv_{i+1}$ in C with $v_{i-2}y_0^+v_iv_{i+1}$, we have a hamiltonian cycle of G . This is a contradiction, and the lemma follows. \square

Win [10] introduced a *k-ended system* to prove the existence of a spanning tree with at most k leaves. In this paper, we modify the definition of a *k-ended system* and define a *k-extended system*. It plays an important role in the proof of our main theorem.

Let G be a connected claw-free graph, and F be a subgraph of G . The set of components of F is denoted by $\mathcal{C}(F)$. We call F an *extended system* if each component of F is a path, a cycle or a double complete graph. For an extended system F , we define a mapping f from $\mathcal{C}(F)$ to $\{1, 2\}$ as follows.

For every $C \in \mathcal{C}(F)$,

$$f(C) = \begin{cases} 1 & \text{if } C \text{ is } K_1, K_2, \text{ a cycle or a double complete graph,} \\ 2 & \text{otherwise (i.e., a path of order at least four),} \end{cases}$$

and define

$$f(F) = \sum_{C \in \mathcal{C}(F)} f(C).$$

Let $\mathcal{C}_i(F) = \{C \in \mathcal{C}(F) : f(C) = i\}$ for $i = 1, 2$. An extended system F is called a k -extended system if $f(F) \leq k$.

The following lemma is an easy but important observation.

Lemma 5 *Let G be a claw-free graph and D be an induced double complete subgraph of G . If a vertex $v \in V(G) - V(D)$ is adjacent to the center of D , then v is also adjacent to a non-central vertex of D .*

Proof. Let D_1 and D_2 be the two blocks of D . Then both D_1 and D_2 are complete graphs. Let x be the center of D and let $x_i \in D_i - \{x\}$ ($i = 1, 2$). Since D is an induced subgraph of G , $x_1x_2 \notin E(G)$. Since $\{x_1, x_2, v\} \subset N_G(x)$ and G is claw-free, $\{x_1v, x_2v\} \cap E(G) \neq \emptyset$. \square

The next lemma shows a relationship between a k -extended system and a spanning tree with at most k leaves in a claw-free graph.

Lemma 6 *Let $k \geq 2$ be an integer and G be a connected claw-free graph. If G has an extended system F_0 , then G has a spanning tree with at most $f(F_0)$ leaves. In particular, if G has a k -extended system, then G has a spanning tree with at most k leaves.*

Proof. Take a spanning extended system F with $f(F) \leq f(F_0)$ so that the number of double complete graphs is as small as possible. Then every double complete graph of F is an induced subgraph of G since if two non-central vertices of a double complete graph D of F are joined by an edge e of G , then $D + e$ has a hamiltonian cycle, and so D should be replaced by this hamiltonian cycle.

Since G is connected, there exists a minimal set X of edges such that F together with X forms a connected spanning subgraph of G . We shall construct a spanning tree with at most k leaves consisting of F and X . By Lemma 5, we may assume that no edge in X is incident with the center of a double complete graph. For any double complete graph D of F , there exists an edge $e_D \in X$ incident with a vertex v_D of D , where v_D is not the center of D . Then D has a hamiltonian path starting at v_D , and we replace D with this hamiltonian path.

For any cycle C of F , there exists an edge $e_C \in X$ incident with a vertex v_C of C . Delete an edge of C incident with v_C . By repeating the above procedure for every double complete graph and every cycle of F , we obtain a spanning tree T . By the construction, for each $C \in \mathcal{C}(F)$, the number of leaves of T contained in C is at most $f(C)$.

Hence T has at most $f(F) \leq f(F_0)$ leaves. \square

We call a k -extended system F of G a *maximal k -extended system* if G has no k -extended system F' such that $V(F)$ is a proper subset of $V(F')$. In order to prove our theorem, we need the following lemma.

Lemma 7 *Suppose that a graph G does not have a spanning k -extended system. Let F be a maximal $(k + 1)$ -extended system of G . Then G does not have a k -extended system F' with $V(F') = V(F)$. In particular, F is not a k -extended system, and so $f(F) = k + 1$.*

Proof. Let F be a maximal $(k + 1)$ -extended system of G . Assume that G has a k -extended system F' with $V(F') = V(F)$. Since G does not have a spanning k -extended system, there exists a vertex $v \in V(G) - V(F')$, and thus G has a $(k + 1)$ -extended system $F' \cup \{v\}$, which contradicts the maximality of F . \square

By Lemma 6, in order to prove our Theorem 3, it suffices to prove the following theorem.

Theorem 8 *Let $k \geq 2$ be an integer and G be a claw-free graph. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning k -extended system.*

Proof. Suppose that G has no spanning k -extended system. Take a maximal $(k + 1)$ -extended system F so that

(F1) $\sum_{P \in \mathcal{C}_2(F)} |P|$ is as large as possible,

(F2) The number of cycles in $\mathcal{C}_1(F)$ is as large as possible subject to (F1), and

(F3) $\sum_{P \in \mathcal{C}_2(F)} (\deg_{\langle V(P) \rangle_G}(x_P) + \deg_{\langle V(P) \rangle_G}(y_P))$ is as small as possible, subject to (F1) and (F2), where x_P and y_P are the end-vertices of P .

By Lemma 7, $f(F) = k + 1$. We begin with a simple but important observation.

Claim 1 *For each $D \in \mathcal{C}_1(F)$ and for each $v \in V(D)$ that is not the center of D if D is a double complete graph, D has a hamiltonian path containing v as one of its end-vertices.*

The next claim follows from the condition (F2) and the same argument as in the first paragraph of the proof of Lemma 6.

Claim 2 *Every double complete graph D of F is an induced subgraph of G .*

Next, we investigate the adjacency between the components of F .

Claim 3 *The following three statements hold.*

- (i) *No two components of $\mathcal{C}_1(F)$ are connected by an edge of G .*
- (ii) *No end-vertex of a path in $\mathcal{C}_2(F)$ is connected to a component of $\mathcal{C}_1(F)$ by an edge of G .*
- (iii) *No two end-vertices of two distinct paths or of the same path in $\mathcal{C}_2(F)$ are joined by an edge of G .*

Proof. (i) Assume that two components Q_1 and Q_2 of $\mathcal{C}_1(F)$ are joined by an edge e of G . By Lemma 5, we may assume that no end-vertex of e is the center of a double complete graph. So $Q_1 + e + Q_2$ contains a hamiltonian path P_0 . By replacing Q_1 and Q_2 of F by P_0 , we obtain another maximal

$(k + 1)$ -extended system F' on $V(F)$. If $|P_0| \geq 4$ this contradicts the condition (F1). If $|P_0| \leq 3$, then $f(P_0) = 1$ and hence F' is a k -extended system, which contradicts Lemma 7.

(ii) If an end-vertex of a path $P \in \mathcal{C}_2(F)$ is joined to a component $Q \in \mathcal{C}_1(F)$ by an edge e of G , then by an argument similar to the one in (i), we see that $P + e + Q$ has a hamiltonian path. Thus, we can derive a contradiction by Lemma 7.

(iii) If two end-vertices of two paths or of the same path in $\mathcal{C}_2(F)$ are joined by an edge of G , then we can obtain a k -extended system with vertex set $V(F)$, which contradicts Lemma 7. \square

For every component $Q \in \mathcal{C}_1(F)$, we take one vertex x_Q from Q so that x_Q is a non-central vertex of Q if Q is a double complete graph. For every path $P \in \mathcal{C}_2(F)$, let x_P and y_P be the two end-vertices of P . Define $\text{End}(F)$ by

$$\text{End}(F) = \bigcup_{Q \in \mathcal{C}_1(F)} \{x_Q\} \cup \bigcup_{P \in \mathcal{C}_2(F)} \{x_P, y_P\}.$$

Then $|\text{End}(F)| = f(F) = k + 1$ by Lemma 7. Claim 3 and Lemma 5 yield the next two claims.

Claim 4 $\text{End}(F)$ is an independent set of G .

Claim 5 For every component $Q \in \mathcal{C}_1(F)$ of F and the vertex $\{x_Q\} = \text{End}(F) \cap V(Q)$, it follows that

$$\sum_{x \in \text{End}(F)} |N_G(x) \cap V(Q)| = |N_G(x_Q) \cap V(Q)| \leq |Q| - 1 = |Q| - f(Q).$$

Now we measure the neighborhood of $\text{End}(F)$ in a path of $\mathcal{C}_2(F)$.

Claim 6 Let P be a path in $\mathcal{C}_2(F)$. Then for each distinct pair of vertices z, w in $\text{End}(F) - \{x_P, y_P\}$, the following statements hold.

- (i) $N_G(z) \cap N_G(w) \cap V(P) = \emptyset$.
- (ii) $N_G(x_P)^- \cap N_G(y_P) \cap V(P) = \emptyset$.
- (iii) $N_G(z)^- \cap N_G(y_P) \cap V(P) = \emptyset$ and $N_G(z)^+ \cap N_G(x_P) \cap V(P) = \emptyset$.
- (iv) $N_G(z) \cap N_G(x_P) \cap V(P) = \emptyset$.

Proof. Let Q and R be the components of F containing z and w , respectively.

(i) Suppose $N_G(z) \cap N_G(w) \cap V(P) \neq \emptyset$ and take a vertex $v \in N_G(z) \cap N_G(w) \cap V(P)$. Then $v \neq x_P, y_P$ by Claim 4. Since $\{z, w, v^-\} \subset N_G(v)$ and G is claw-free, $zv^- \in E(G)$ or $wv^- \in E(G)$. By symmetry, we may assume that $zv^- \in E(G)$. If $Q \neq R$, then replace P, Q, R of F by two hamiltonian paths Q' and R' in $P(x_P, v^-) + v^-z + Q$ and $P(y_P, v) + vw + R$, respectively. Then we obtain a new $(k+1)$ -extended system F' on $V(F)$. If $f(Q') + f(R') < f(P) + f(Q) + f(R)$, then F' is a k -extended system, which contradicts Lemma 7. Thus, $f(Q') + f(R') \geq f(P) + f(Q) + f(R)$. This is possible only if $\{Q', R'\} \subset \mathcal{C}_2(F')$ and $\{Q, R\} \subset \mathcal{C}_1(F)$. However, this contradicts the condition (F1). If $Q = R$, then Q is a path whose end-vertices are z and w and $P(x_P, v^-) + v^-z + Q + vw + P(v, y_P)$ is a hamiltonian path of $\langle V(P) \cup V(Q) \rangle_G$, and by replacing P and Q with this path, we have a k -extended system on $V(F)$, contradicting Lemma 7.

(ii) If $N_G(x_P)^- \cap N_G(y_P) \cap V(P) \neq \emptyset$, then $\langle V(P) \rangle_G$ has a hamiltonian cycle, and so G has a k -extended system with vertex set $V(F)$, which contradicts Lemma 7.

(iii) By symmetry, it suffices to show that $N_G(z)^- \cap N_G(y_P) \cap V(P) = \emptyset$. Assume that there exists a vertex $v \in N_G(z)^- \cap N_G(y_P) \cap V(P)$. Then $P(x_P, v) + vy_P + P(y_P, v^+) + v^+z + Q$ has a hamiltonian path of $\langle V(P) \cup V(Q) \rangle_G$, and so by replacing P and Q of F with this path, we have a k -extended system on $V(F)$. This contradicts Lemma 7.

(iv) Suppose that there exists a vertex v in $N_G(z) \cap N_G(x_P) \cap V(P)$. Then $v \neq y_P$ by Claim 4. Since $\{v^+, x_P, z\} \subset N_G(v)$ and G is claw-free, we have $v^+z \in E(G)$ by (iii) and Claim 4. Suppose that Q is a path of order at least four. If $v \neq x_P^+$, then replace P and Q by the cycle $P(x_P, v) + vx_P$ and a hamiltonian path of $P(y_P, v^+) + v^+z + Q$. If $v = x_P^+$, replace P and Q with x_Pv and a hamiltonian path of $P(y_P, v^+) + v^+z + Q$. In either case, G has a k -extended system on $V(F)$, which contradicts Lemma 7.

Next suppose that Q is a cycle. Let us denote the two vertices of Q adjacent to z by z^- and z^+ . Then since $\{v, z^-, z^+\} \subset N_G(z)$ and

G is claw-free, we may assume that $z^-v \in E(G)$ or $z^-z^+ \in E(G)$ by symmetry. If $z^-v \in E(G)$, then $P(x_P, v) + vz^- + Q + zv^+ + P(v^+, y_P)$ has a hamiltonian path, and by replacing P and Q with this path, we again have a k -extended system on $V(F)$, a contradiction. Therefore we may assume that $z^-z^+ \in E(G)$. If the order of Q is at least four, replace P and Q with the path $P' = P(x_P, v) + vz + zv^+ + P(v^+, y_P)$ and the cycle $Q - z + z^-z^+$. If the order of Q is three, replace P and Q with the path P' and z^-z^+ . Then in either case, we obtain a maximal $(k + 1)$ -extended system with $\sum_{P \in \mathcal{C}_2(F')} |P| > \sum_{P \in \mathcal{C}_2(F)} |P|$. This contradicts the condition (F1).

We finally consider the case that Q is K_1 , K_2 or a double complete graph. In this case, consider $Q - z$ and the path $P' = P(x_P, v) + vz + zv^+ + P(v^+, y_P)$. Note that $Q - z$ is empty, K_1 , K_2 , a double complete graph or a complete graph of order at least three. In the last case, $Q - z$ has a hamiltonian cycle. Therefore, by replacing P and Q with P' and a certain subgraph of $Q - z$, we obtain a maximal $(k + 1)$ -extended system F' with $\sum_{P \in \mathcal{C}_2(F')} |P| > \sum_{P \in \mathcal{C}_2(F)} |P|$. This contradicts the choice (F1) of F . \square

Claim 7 For each $P \in \mathcal{C}_2(F)$,

$$\sum_{x \in \text{End}(F)} |N_G(x) \cap V(P)| \leq |P| - f(P).$$

Proof. First assume that $N_G(z) \cap V(P) = \emptyset$ for every $z \in \text{End}(F) - \{x_P, y_P\}$. Let $H = \langle V(P) \rangle_G$. By the condition (F3), for each hamiltonian path P^* of H ,

$$\begin{aligned} \sum_{Q \in \mathcal{C}_2(F) - \{P\}} (\deg_{\langle V(Q) \rangle_G}(x_Q) + \deg_{\langle V(Q) \rangle_G}(y_Q)) + \deg_H(x_{P^*}) + \deg_H(y_{P^*}) \\ \geq \sum_{Q \in \mathcal{C}_2(F)} (\deg_{\langle Q \rangle_G}(x_Q) + \deg_{\langle Q \rangle_G}(y_Q)), \end{aligned}$$

which implies $\deg_H(x_{P^*}) + \deg_H(y_{P^*}) \geq \deg_H(x_P) + \deg_H(y_P)$. Thus, if $\deg_H(x_P) + \deg_H(y_P) \geq |H| - 1$, then by Lemma 4, either H has a hamiltonian cycle or H is a double complete graph. Then whichever occurs,

we can replace P with an appropriate subgraph of H to obtain a k -extended system on $V(F)$, which contradicts Lemma 7. Therefore,

$$\begin{aligned} \sum_{x \in \text{End}(F)} |N_G(x) \cap V(P)| &= |N_G(x_P) \cap V(P)| + |N_G(y_P) \cap V(P)| \\ &= \deg_H(x_P) + \deg_H(y_P) \leq |H| - 2 = |P| - f(P). \end{aligned}$$

Next we assume that $N_G(z_1) \cap V(P) \neq \emptyset$ for some vertex $z_1 \in \text{End}(F) - \{x_P, y_P\}$. Let $v \in N_G(z_1) \cap V(P)$, $P_1 = P(x_P, v^-)$ and $P_2 = P(v^+, y_P)$. Then $|P| = |P_1| + |P_2| + 1$. By Claim 6 (i)–(iv), $(N_G(x_P) \cap V(P_1))^-$, $N_G(y_P) \cap V(P_1)$ and

$$\left((N_G(z) \cap V(P_1))^- \right)_{z \in \text{End}(F) - \{x_P, y_P\}}$$

are well-defined and these $k+1$ sets are pairwise disjoint. Moreover, they do not contain v^- by Claim 6 (iii). Thus

$$\sum_{z \in \text{End}(F)} |N_G(z) \cap V(P_1)| \leq |P_1| - 1.$$

By symmetry of P_1 and P_2 , we obtain $\sum_{z \in \text{End}(F)} |N_G(z) \cap V(P_2)| \leq |P_2| - 1$. By Claim 6 (i) and (iv), v is not adjacent to any vertex in $\text{End}(F) - \{z_1\}$, and so $\sum_{z \in \text{End}(F)} |N_G(z) \cap \{v\}| = 1$. By summing these three inequalities, we have

$$\begin{aligned} \sum_{z \in \text{End}(F)} |N_G(z) \cap V(P)| &= \sum_{z \in \text{End}(F)} |N_G(z) \cap V(P_1)| + \sum_{z \in \text{End}(F)} |N_G(z) \cap V(P_2)| \\ &\quad + \sum_{z \in \text{End}(F)} |N_G(z) \cap \{v\}| \\ &\leq |P_1| - 1 + |P_2| - 1 + 1 \\ &= |P| - 2 = |P| - f(P). \quad \square \end{aligned}$$

We now prove Theorem 8. Assume that $N_G(z) \cap N_G(w) - V(F) \neq \emptyset$ for some $z, w \in \text{End}(F)$ with $z \neq w$. Let P and Q be the components of F that contain z and w , respectively (possibly $P=Q$). Let $a \in N_G(z) \cap N_G(w) - V(F)$. If $P \neq Q$, then since P and Q have hamiltonian paths which contain z and w as an end-vertex, respectively, $P + za + aw + Q$

contains a hamiltonian path. By replacing P and Q with this path, we obtain a new $(k + 1)$ -extended system F' with $V(F') = V(F) \cup \{a\}$. This contradicts the maximality of F . If $P = Q$, then we may assume $z = x_{P_0}$ and $w = y_{P_0}$ for some P_0 . Then by replacing P with a cycle $P + az + zw$, we again obtain a $(k + 1)$ -extended system F' with $V(F') = V(F) \cup \{a\}$, a contradiction. Therefore, we have $N_G(z) \cap N_G(w) - V(F) = \emptyset$ for each distinct pair of vertices z and w in $\text{End}(F)$. Hence

$$\sum_{z \in \text{End}(F)} |N_G(z) \cap (V(G) - V(F))| \leq |V(G) - V(F)| = |G| - |F|.$$

Then by Claims 5 and 7, we obtain

$$\begin{aligned} \sum_{z \in \text{End}(F)} \deg_G(z) &= \sum_{C \in \mathcal{C}(F)} \sum_{z \in \text{End}(F)} |N_G(z) \cap V(C)| \\ &\quad + \sum_{z \in \text{End}(F)} |N_G(z) \cap (V(G) - V(F))| \\ &\leq \sum_{C \in \mathcal{C}(F)} (|C| - f(C)) + |G| - |F| \\ &= |F| - f(F) + |G| - |F| \\ &= |G| - k - 1. \end{aligned}$$

This contradicts the condition $\sigma_{k+1}(G) \geq |G| - k$, and Theorem 8 follows.

□

3 Maximum Degree

A tree of maximum degree at most k is called a k -tree. Under the same assumption as that of Theorem 3, we can actually guarantee the existence of a 3-tree with at most k leaves.

Theorem 9 *Let $k \geq 2$ be an integer and let G be a connected claw-free graph. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning 3-tree with at most k leaves.*

In order to prove the above theorem, it suffices to prove the following lemma.

Lemma 10 *Let $k \geq 2$ be an integer. If a connected claw-free graph G has a spanning tree with at most k leaves, then G has a spanning 3-tree with at most k leaves.*

Proof. Let u be an arbitrary vertex in G , and consider every spanning tree as a rooted tree with root u . Choose a spanning tree T with at most k leaves so that $\sum_{x \in V(T)} \text{dist}_T(u, x)$ is as large as possible, where $\text{dist}_T(x, y)$ is the distance in T between two vertices x and y . Assume T has a vertex w of degree at least four. Then w has at least three children, and since G is claw-free, w has a pair of children v_1 and v_2 which are adjacent with each other in G . Let $T' = T - wv_1 + v_1v_2$. Then T' is a spanning tree of G , and $\text{deg}_{T'}(w) = \text{deg}_T(w) - 1$, $\text{deg}_{T'}(v_2) = \text{deg}_T(v_2) + 1$ and $\text{deg}_{T'}(x) = \text{deg}_T(x)$ for each $x \in V(G) - \{w, v_2\}$. Since $\text{deg}_T(w) \geq 4$, T' does not have the larger number of leaves than T .

Let $x \in V(G)$. Then T has a unique ux -path P . If P still exists in T' , we have $\text{dist}_T(u, x) = \text{dist}_{T'}(u, x)$. If P does not exist in T' , then $wv_1 \in E(P)$ and $P' = P(u, w) + wv_2 + v_2v_1 + P(v_1, x)$ is a unique ux -path in T' . This implies $\text{dist}_{T'}(u, x) = \text{dist}_T(u, x) + 1$. Therefore, $\text{dist}_{T'}(u, x) \geq \text{dist}_T(u, x)$ for each $x \in V(G)$ and $\text{dist}_{T'}(u, v) > \text{dist}_T(u, v)$. These imply $\sum_{x \in V(G)} \text{dist}_{T'}(u, x) > \sum_{x \in V(G)} \text{dist}_T(u, x)$. This contradicts the choice of T , and hence we have $\Delta(T) \leq 3$ \square

4 Concluding Remarks

Matthews and Sumner [5] proved that a 2-connected claw-free graph of minimum degree at least $\frac{1}{3}(|G| - 2)$ has a hamiltonian cycle. This result was later extended by Zhang [11].

Theorem 11 (Zhang [11]) *A k -connected claw-free graph G with $\sigma_{k+1}(G) \geq |G| - k$ has a hamiltonian cycle.*

Interpreting a hamiltonian cycle as a “spanning tree with one leaf” and comparing Theorems 3 and 11, we may make the following conjecture.

Conjecture 12 *For integers k and m with $k \geq 2$ and $m \leq \min\{6, k - 1\}$, every m -connected claw-free graph G with $\sigma_{k+1}(G) \geq |G| - k$ has a spanning tree with at most $k - m + 1$ leaves.*

The assumption $m \leq 6$ in the above conjecture looks strange, but it comes from the following theorem by Ryjáček [7].

Theorem 13 (Ryjáček [7]) *Every 7-connected claw-free graph is hamiltonian.*

By the above theorem, a 7-connected claw-free graph has a spanning tree with two leaves without any degree sum condition.

References

- [1] H. Broersma and H. Tuinstra, Independence trees and Hamilton cycles, *J. Graph Theory* **29** (1998), 227–237.
- [2] Y. Egawa, H. Matsuda, T. Yamashita and K. Yoshimoto, On a spanning tree with specified leaves, *Graphs Combin.* **24** (2008), 13–18.
- [3] H. Enomoto, Long paths and large cycles in finite graphs, *J. Graph Theory* **8** (1984), 287–301.
- [4] H. A. Jung, On maximal circuits in finite graphs, *Ann. Discrete Math.* **3** (1978), 129–144.
- [5] M. M. Matthews and D. P. Sumner, Longest paths and cycles in $K_{1,3}$ -free graphs. *J. Graph Theory* **9** (1985), 269–277.
- [6] C. Nara, On sufficient conditions for a graph to be hamiltonian, *Natur. Sci. Rep. Ochanomizu Univ.* **31** (1980), 75–80.
- [7] Z. Ryjáček, On a closure concept in claw-free graphs, *J. Combin. Theory Ser. B* **70** (1997), 217–224.

- [8] M. Tsugaki and T. Yamashita, Spanning trees with few leaves, *Graphs Combin.* **23** (2007), 585–598.
- [9] D.B. West, *Introduction to Graph Theory*, Second Edition, Prentice-Hall, Inc. 1996.
- [10] S. Win, On a conjecture of Las Vergnas concerning certain spanning trees in graphs, *Resultane Math.* **2** (1979), 215–224.
- [11] C.Q. Zhang, Hamilton cycles in claw-free graphs, *J. Graph Theory* **12** (1988), 209–216.