The independence number condition for the existence of a spanning f-tree

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Abstract

Let G be a graph and f be a mapping from $V(G)$ to the positive integers. A subgraph T of G is called an *f-tree* if T forms a tree and $d_T(x) \leq f(x)$ for any $x \in V(T)$. We propose a conjecture on the existence of a spanning f-tree, and give a partial solution to it.

1 Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. We denote the degree of a vertex x by $d_G(x)$. Let $\alpha(G)$ and $\kappa(G)$ be the independence number and the connectivity of a graph G, respectively.

For a graph G, let f be a mapping from $V(G)$ to the positive integers. An *f-tree* T is defined as a subgraph of G which forms a tree such that $d_T(v) \leq f(v)$ for any $v \in V(T)$. When $V(T) = V(G)$, T is called a *spanning f-tree*. In this paper, we concentrate on the existence of a spanning f-tree. When f is a constant mapping taking the value k, an f-tree T is called a *k-tree*. Neumann-Lara and Rivera-Campo showed the following result on the existence of a spanning k-tree.

Theorem 1 (Neumann-Lara and Rivera-Campo [3]) *Let k and n be integers with $k \geq 3$, and let G be an n-connected graph. If $\alpha(G) \leq n(k-1)+1$, then there exists a spanning k-tree of G.*

Matsuda and Matsumura gave the following result on the existence of a spanning k-tree with specified leaves, which is an extension of Theorem 1.

Theorem 2 (Matsuda and Matsumura [2]) *Let n, k and s be integers with $k \geq 2$, $0 \leq s \leq k$ and $s+1 \leq n$, and let G be an n-connected graph. If $\alpha(G) \leq (n-s)(k-1)+1$, then for any s vertices of G, G has a spanning k-tree such that the s specified vertices are contained in the set of leaves.*

Extending this result to an f-tree, we propose the following conjecture.

Conjecture 3 *Let n be an integer, G be an n-connected graph and f be a mapping from $V(G)$ to the positive integers. If $\sum_{x\in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G) \leq \min_R \left\{ \sum_{x\in R}(f(x)-1) : R \subset V(G), |R| = n \right\} + 1$, then there exists a spanning f-tree of G.*

Suppose that there exists a spanning f-tree T. Then

$$\begin{aligned} \sum_{x\in V(G)} f(x) &\geq \sum_{x\in V(G)} d_T(x) \\ &= 2|E(T)| \\ &= 2(|V(G)|-1). \end{aligned}$$

Therefore for the existence of a spanning f-tree, the condition "$\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$" is necessary.

On the other hand, the following graph G_1 shows that the independence number condition in Conjecture 3 is sharp. Let S be a clique with $|V(S)| = n$ and f be a mapping from $V(S)$ to the positive integers. Let t denote the value $t = \sum_{x \in S}(f(x)-1)+2$, let l be an arbitrary chosen positive integer, and let "+" denote the join of two graphs. We define $G_1 = S + tK_l$. Extend f to a mapping from $V(G_1)$ such that $f(x) \leq f(y)$ for any $x \in S$ and $y \in V(G) - S$. Then G_1 is n-connected, $\alpha(G_1) = t = \sum_{x \in S}(f(x)-1)+2 = \min_R \{\sum_{x \in R}(f(x)-1) : R \subset V(G_1), |R| = n\} + 2$ and G_1 has no spanning f-tree.

In this paper, we show the following result, which gives a partial solution to Conjecture 3. For a mapping f, let $S_i(f) = \{x \in V(G) : f(x) = i\}$ and $s_i(f) = |S_i(f)|$.

Theorem 4 *Let n be a positive integer, G be an n-connected graph and f be a mapping from $V(G)$ to the positive integers. Suppose $s_1(f) + s_2(f) \leq n+1$, $\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G) \leq \min_R \{\sum_{x \in R}(f(x)-1) : R \subset V(G), |R| = n\} + 1$. Then there exists a spanning f-tree in G.*

Let f_1 be a mapping on $V(G)$ which assigns the value 1 to s given vertices and the value k to the other vertices. Then a spanning f_1-tree is a k-tree as desired in Theorem 2. Moreover,

$$\begin{aligned} &\min_R \left\{ \sum_{x \in R}(f_1(x)-1) : R \subset V(G), |R| = n \right\} + 1 \\ &= s(1-1) + (n-s)(k-1) + 1 \\ &= (n-s)(k-1) + 1, \end{aligned}$$

and hence Theorem 2 is a special case of Conjecture 3. If $k \geq 3$, then $s_1(f_1) + s_2(f_1) = s \leq n+1$. This implies that Theorem 4 is a generalization of Theorem 2 for $k \geq 3$. Note that the essential part of the proof of Theorem 2 is only the case $k \geq 3$, because the case $k = 2$ is contained in the following theorem by Chvátal and Erdős.

Theorem 5 (Chvátal and Erdős [1]) *Let m be an integer and M be an m-connected graph.*

(i) *If $\alpha(M) \leq m + 1$, then there exists a Hamilton path.*

(ii) *If $\alpha(M) \leq m$, then for any vertex u, there exists a Hamilton path with one end u.*

(iii) *If $\alpha(M) \leq m - 1$, then for any two vertices u, v, there exists a Hamilton path connecting u and v.*

Let g be a mapping from $V(G)$ to the positive integers. If $d_T(x) = g(x)$, a vertex x in G is called *g-saturated* in a subgraph T, and let $A_T(g)$ be the set of g-saturated vertices in T. Let $\mathcal{G}$ be the set of graphs obtained from two disjoint complete graphs (of order $l_1, l_2 \geq 2$, respectively) joined by a bridge. (See Figure 1.)

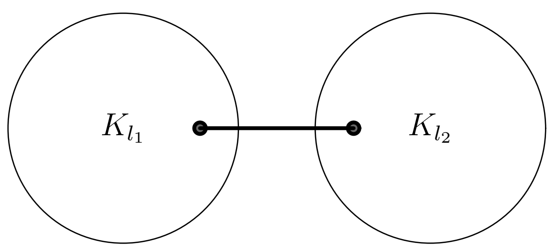

Figure 1: A graph contained in $\mathcal{G}$.

We call a pair (M, S) an *exception pair* if $M \in \mathcal{G}$ and S is the set of which are end-vertices of the bridge in M.

Theorem 6 *Let M be an m-connected graph and g be a mapping from $V(M)$ to the positive integers. Suppose $s_1(g) = 0$, $s_2(g) \leq m+1$, and $\alpha(M) \leq \min_{R'}\{\sum_{x \in R'}(g(x)-1) : R' \subset V(M), |R'| = m\} + 1$. Then there exists a spanning g-tree T such that $|A_T(g)| \leq m$, unless $m = 1$, $s_2(g) = 2$, and $(M, S_2(g))$ is an exception pair.*

Observe that the condition "$s_2(g) \leq m+1$" in Theorem 6 is best possible. Let G_2 be the graph obtained from $K_{m,m+2}$ by inserting one edge into the larger vertex set of the bipartition. We define the mapping g as follows;

$$g(x) = \begin{cases} 3 & \text{if } d_{G_2}(x) = m, \\ 2 & \text{otherwise.} \end{cases}$$

In this graph G_2, we have $\alpha(G_2) = m+1$, $s_2(g) = m+2$ and G_2 has no spanning g-tree T such that $|A_T(g)| \leq m$.

We will prove Theorem 6 by considering two cases. First, we consider the case $s_2(g) \leq m$. In this case, we will show a more general theorem.

Theorem 7 *Let m and c be integers, M be an m-connected graph and g be a mapping from $V(M)$ to the positive integers. Suppose $s_1(g) = 0$ and $s_2(g) \leq c \leq m$. If $\alpha(M) \leq \min_{R'}\{\sum_{x \in R'}(g(x)-2) : R' \subset V(M), |R'| = m\} + c + 1$, then there exists a spanning g-tree T such that $|A_T(g)| \leq c$.*

This is a generalization of the following theorem on k-trees, which was showen by Neumann-Lara and Rivera-Campo for $k \geq 4$ and by Tsugaki for $k = 3$.

Theorem 8 (Neumann-Lara and Rivera-Campo [3], Tsugaki [4]) *Let k, m and c be integers with $k \geq 3$ and $c \leq m$, and let M be an m-connected graph. If $\alpha(M) \leq m(k-2) + c + 1$, then there exists a spanning k-tree T such that the number of vertices which have degree k in T is at most c.*

Secondly, we consider the case $s_2(g) = m + 1$. To prove this case, we use the following theorem.

Theorem 9 *Let m be an integer and M be an m-connected graph. If $\alpha(M) \leq m + 1$, then for every $S \subset V(M)$ with $|S| = m + 1$, there exists a Hamilton path P such that at least one of the end-vertices of P is contained in S, unless $m = 1$ and (M, S) is an exception pair.*

Observe that Theorem 9 is an extension of Theorem 5 (i).

2 Proofs

Before proving Theorem 7, we need some definitions.

Let G be a graph. In [5], Win defined a *path and cycle system* as a spanning subgraph in which each component is a path or a cycle. More generally, we define a *tree and cycle system* as a spanning subgraph in which each component is a tree or a cycle. For a mapping g from $V(G)$ to the positive integers, a tree and cycle system F is called a *g-system* if $d_F(x) \leq g(x)$ for any $x \in V(F)$.

For a tree or a cycle C, define

$$h(C) = \begin{cases} 1 & \text{if } C \text{ is a cycle, or a path which has at most 2 vertices,} \\ 2 & \text{otherwise,} \end{cases}$$

and for a tree and cycle system F with p components, $F = C_1 \cup C_2 \cup \cdots \cup C_p$, define

$$h(F) = \sum_{i=1}^{p} h(C_i).$$

We prove Theorem 7 using Win's theorem.

Theorem 10 (Win [5]) *Let m be an integer and M be an m-connected graph. If $\alpha(M) \leq m + l + 1$, there exists a path and cycle system W with $h(W) \leq l + 2$.*

Proof of Theorem 7. Let $S_2 = S_2(g)$ and $s_2 = s_2(g)$. By Theorem 10, there exists a path and cycle system W with

$$h(W) \leq \min_{R'} \left\{ \sum_{x \in R'} (g(x) - 2) : R' \subset V(M), |R'| = m \right\} - m + c + 2. \qquad (1)$$

It is sufficient to prove the following claim. For a graph G, let $\omega(G)$ be the number of components of G and let $B_F(g) = \{x \in A_F(g) : g(x) \geq 3\}$.

Claim 1 *For every i with $1 \leq i \leq \omega(W)$, there exists a g-system F such that*

(i) $\omega(F) = \omega(W) - i + 1$,

(ii) $\sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \leq h(W) - h(F)$,

(iii) $|B_F(g)| \leq c - s_2$.

Proof. We prove Claim 1 by induction on i. Suppose that $i = 1$. Since each component of W is a path or a cycle, we have $d_W(x) \leq 2 \leq g(x)$, and hence W is a g-system. Moreover, (i) $\omega(W) = \omega(W) - i + 1$, (ii) $\sum_{x \in V(M)} \max\{d_W(x) - 2, 0\} = 0 = h(W) - h(W)$, and (iii) $|B_W(g)| = 0 \leq c - s_2$. Thus, W is a desired g-system.

Suppose that $i \geq 2$. By the induction hypothesis, there exists a g-system F' such that

(i) $\omega(F') = \omega(W) - (i - 1) + 1 = \omega(W) - i + 2$,

(ii) $\sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} \leq h(W) - h(F')$,

(iii) $|B_{F'}(g)| \leq c - s_2$.

Subclaim We may assume that $|B_{F'}(g)| < m - s_2$.

Proof. Assume that $|B_{F'}(g)| \geq m - s_2$. By the condition (iii), we have $m = c$ and $|B_{F'}(g)| = m - s_2$. Let $L = B_{F'}(g) \cup S_2$. Note that $|L| = m$. Then by the definition of $B_{F'}(g)$, we have

$$\begin{aligned}
&\sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} \\
&\geq \sum_{x \in L} \max\{d_{F'}(x) - 2, 0\} \\
&= \sum_{x \in L} (g(x) - 2). \qquad (2)
\end{aligned}$$

Thus, by the condition (ii) and the inequalities (1) and (2),

$$\begin{aligned}
&\sum_{x \in L} (g(x) - 2) \\
&\leq \sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} \\
&\leq h(W) - h(F') \\
&\leq \min_{R'} \left\{ \sum_{x \in R'} (g(x) - 2) : R' \subset V(M), |R'| = m \right\} - m + c + 2 - h(F') \\
&\leq \sum_{x \in L} (g(x) - 2) + 2 - h(F'),
\end{aligned}$$

or

$$h(F') \leq 2.$$

On the other hand, by the condition (i), we have $\omega(F') = \omega(W) - i + 2 \geq 2$ and hence $h(F') \geq \sum_{C \text{ is a component of } F} h(C) \geq \omega(F') \geq 2$. Then

$$h(F') = \omega(F') = 2.$$

This implies that each component of F' is a cycle or a path of order at most 2, and hence $|B_{F'}(g)| = 0$. Since $|B_{F'}(g)| = m - s_2$, we have $m = s_2$. By the assumption of Theorem 7, we have $\alpha(M) \leq m + 1$. Then by Theorem 5 (i), there exists a Hamilton path P in M and P is a desired g-system. □

Choose $L \subset V(M)$ so that $S_2 \subset L$, $|L| = m-1$ and $\sum_{v \in L}(g(v) - d_{F'}(v))$ is as small as possible. Note that $A_{F'}(g) \subset L$ by the Subclaim and by the choice of L.

We consider two cases. For $y \in V(M)$, let C_y be the component of F' such that $y \in V(C_y)$. When C_y is a cycle, let e_y be an edge of $E(C_y)$ such that e_y is incident to y.

Case 1. $M - L$ is formed by vertices of at least two components of F'.

In this case, since M is m-connected and $|L| = m-1$, there exists $yz \in E(M)$ such that $y, z \notin L$ and $C_y \neq C_z$. By the symmetry, we may assume that $g(z) - d_{F'}(z) \leq g(y) - d_{F'}(y)$.

Case 2. $M - L$ is formed by vertices of one component of F'.

Let C be the unique component of F' such that $(V(M) - L) \cap V(C) \neq \emptyset$. We take $y \in V(M-C)$ so that y is a leaf of C_y, if possible. By the assumption of Case 2, note that $y \in L$. Since F' is a tree and cycle system, if y is not a leaf of C_y, then C_y must be a cycle. Because M is m-connected, there exists $z \in V(C) - L$ such that $yz \in E(M)$.

In both cases, we define

$$F = \begin{cases} (F' - \{e_y\}) \cup \{yz\} & \text{if } C_y \text{ is a cycle and } C_z \text{ is not a cycle,} \\ (F' - \{e_z\}) \cup \{yz\} & \text{if } C_y \text{ is not a cycle and } C_z \text{ is a cycle,} \\ (F' - \{e_y, e_z\}) \cup \{yz\} & \text{if both } C_y \text{ and } C_z \text{ are cycles,} \\ F' \cup \{yz\} & \text{otherwise,} \end{cases}$$

and let C_{yz} be the component of F that contains y and z. Then $d_F(x) = d_{F'}(x) \leq g(x)$ for every $x \in V(M) - \{y, z\}$. Since $z \notin A_{F'}(g) \subset L$, we have $d_F(z) \leq d_{F'}(z) + 1 \leq g(z)$. By the definition of y, $d_F(y) \leq d_{F'}(y) + 1 \leq g(y)$ in Case 1 and $d_F(y) \leq 2 \leq g(y)$ in Case 2. Therefore F is a g-system. Now, we will check that F is a desired g-system.

(i) $\omega(F) = \omega(F') - 1 = \omega(W) - i + 1$, and hence F satisfies the condition (i).

(ii) For every $x \in V(M) - \{y, z\}$, we have $d_F(x) = d_{F'}(x)$, in particular, $\max\{d_F(x) - 2, 0\} = \max\{d_{F'}(x) - 2, 0\}$.

By the definition of F, if $h(C_y) = 1$, then $\max\{d_F(y) - 2, 0\} = \max\{d_{F'}(y) - 2, 0\} = 0$, and if $h(C_y) = 2$, then $d_F(y) \leq d_{F'}(y) + 1$. Thus, in both cases, $\max\{d_F(y) - 2, 0\} \leq \max\{d_{F'}(y) - 2, 0\} + h(C_y) - 1$. By the same way, $\max\{d_F(z) - 2, 0\} \leq \max\{d_{F'}(z) - 2, 0\} + h(C_z) - 1$.

Since $h(F) = h(F') - h(C_y) - h(C_z) + h(C_{yz}) \leq h(F') - h(C_y) - h(C_z) + 2$, we have

$$h(F') \geq h(F) + h(C_y) + h(C_z) - 2.$$

Therefore,

$$\begin{aligned}
&\sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \\
\leq\ & \sum_{x \in V(M)} \max\{d_{F'}(x) - 2, 0\} + h(C_y) + h(C_z) - 2 \\
\leq\ & h(W) - h(F') + h(C_y) + h(C_z) - 2 \\
\leq\ & h(W) - h(F).
\end{aligned}$$

Thus, F satisfies the condition (ii).

(iii) Assume that $|B_F(g)| > |B_{F'}(g)|$. Then by the definition of y and z, $z \in B_F(g) - B_{F'}(g)$ and hence $g(z) - d_{F'}(z) = 1$. By the definition of L, we obtain $d_F(x) \geq d_{F'}(x) \geq g(x) - 1$ for any $x \in L - S_2$. Let $L' = L \cup \{z\}$. Then we have $\max\{d_F(x) - 2, 0\} \geq g(x) - 3$ for every $x \in L' - B_F(g)$ and $\max\{d_F(x) - 2, 0\} = g(x) - 2$ for every $x \in B_F(g)$ by the definition of L and $B_F(g)$. Let $\varepsilon = 1$ if $y \in B_F(g) - B_{F'}(g)$; otherwise $\varepsilon = 0$. Since

$B_F(g) \cup S_2 \subset L' \cup \{y\}$,

$$\begin{aligned}
&\sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \\
\geq\ & \sum_{x \in L'} \max\{d_F(x) - 2, 0\} + \varepsilon \\
\geq\ & \sum_{x \in L' - (B_F(g) \cup S_2)} (g(x) - 3) + \sum_{x \in L' \cap (B_F(g) \cup S_2)} (g(x) - 2) + \varepsilon \\
=\ & \sum_{x \in L'} (g(x) - 3) + |B_F(g)| + |S_2| \\
=\ & \sum_{x \in L'} (g(x) - 2) - m + |B_F(g)| + s_2. \qquad (3)
\end{aligned}$$

On the other hand, by the inequality (1),

$$\begin{aligned}
& h(W) - h(F) \\
\leq\ & \min_{R'} \Big\{ \sum_{x \in R'} (g(x) - 2) : R' \subset V(M), |R'| = m \Big\} - m + c + 2 - h(F) \\
\leq\ & \sum_{x \in L'} (g(x) - 2) - m + c + 2 - h(F) \qquad (4)
\end{aligned}$$

Thus, by the condition (ii) and the inequalities (3) and (4),

$$\begin{aligned}
& \sum_{x \in L'} (g(x) - 2) - m + |B_F(g)| + s_2 \\
\leq\ & \sum_{x \in V(M)} \max\{d_F(x) - 2, 0\} \\
\leq\ & h(W) - h(F) \\
\leq\ & \sum_{x \in L'} (g(x) - 2) - m + c + 2 - h(F),
\end{aligned}$$

or

$$|B_F(g)| \leq c + 2 - h(F) - s_2.$$

Because $h(F) \geq 2$, this implies that

$$|B_F(g)| \leq c - s_2.$$

Therefore F satisfies the condition (iii) and this completes the proof of Claim 1 and Theorem 7. $\square$

Proof of Theorem 9.

Let M be a graph and S be a subset of $V(M)$ satisfying the assumption of Theorem 9 and (M, S) is not an exception pair. Let $\mathcal{P}$ be the set of paths such that at least one of the end-vertices is contained in S, and let P be a longest path in $\mathcal{P}$. If P contains all vertices of M, then there is nothing to prove. Therefore we may assume that there exists $x_0 \in V(M - P)$.

We give an orientation to P and write the oriented path P by $\overrightarrow{P}$. Let v be the start vertex and u be the terminal vertex of P, respectively. Since at least one end-vertex of P is contained in S, we may assume that $v \in S$.

For $x, y \in V(P)$, we denote an xy-path on $\overrightarrow{P}$ by $x\overrightarrow{P}y$, and write the reverse seaquence of $x\overrightarrow{P}y$ by $y\overleftarrow{P}x$. For $x \in V(P)$, we denote the successor and the predecessor of x on $\overrightarrow{P}$ by x^+ and x^-, respectively. Note that there exist x^+ and x^- except for u^+ and v^-. For $X \subset V(P)$, we define $X^+ = \{x^+ : x \in X - \{u\}\}$ and $X^- = \{x^- : x \in X - \{v\}\}$.

Since M is m-connected, there exist l $(l \geq m)$ internally disjoint paths $\{Q_1, Q_2, \cdots, Q_l\}$ where Q_i connects x_0 and x_i with $\{x_i\} = V(Q_i \cap P)$. We may assume that $x_1, x_2, \cdots, x_l$ are along on $\overrightarrow{P}$. Let $X = \{x_1, x_2, \cdots, x_l\}$.

The following claim is obvious.

Claim 2 (i) *$x_l \neq u$.*

(ii) *$|X^+| = l$.*

(iii) *If $x_1 = v$, then $|X^-| = l - 1$; otherwise $|X^-| = l$.*

(iv) *$X^+ \cup \{x_0\}$ is an independent set.*

(v) *$X^- \cup \{x_0, u\}$ is an independent set.*

By Claim 2, the following claim is shown.

Claim 3 *$l = m$ and $x_1 = v$.*

Proof. By Claim 2 (iii) and (v), $X^- \cup \{x_0, u\}$ is an independent set of order $l+1$ or $l+2$ depending on $x_1 = v$ or $x_1 \neq v$. Since $l \geq m$ and $\alpha(M) \leq m+1$, we have $l = m$ and $x_1 = v$. $\square$

Claim 4 *$u \notin S$ and $x_0 \notin S$.*

Proof. Assume that $u \in S$ (or $x_0 \in S$). Then by Claim 3, the path $u \overleftarrow{P} x_1 Q_1 x_0$ (or $x_0 Q_1 x_1 \overrightarrow{P} u$, respectively) is contained in $\mathcal{P}$, and longer than P, a contradiction. $\square$

Let K be the graph induced by $V(x_m^+ \overrightarrow{P} u)$.

Claim 5 *K is a complete graph.*

Proof. Suppose that there exist $a, b \in V(K)$ such that $ab \notin E(M)$. Choose such vertices a and b so that $a \overrightarrow{P} b$ is as long as possible.

Since $(X - \{x_m\})^+ \cup \{x_0, a, b\}$ is of order $m+2$, this set is not independent. Since $ab \notin E(G)$, there exists $x_i \in X - \{x_m\}$ such that $x_i^+ a \in E(G)$ or $x_i^+ b \in E(G)$. By the choice of a and b, we have $a^- u \in E(G)$ or $a = x_m^+$, and $x_m^+ b^+ \in E(G)$ or $b = u$. Let

$$P' = \begin{cases} x_1 \overrightarrow{P} x_i Q_i x_0 Q_m x_m \overleftarrow{P} x_i^+ a \overrightarrow{P} u (a^- \overleftarrow{P} x_m^+) & \text{if } x_i^+ a \in E(G), \\ x_1 \overrightarrow{P} x_i Q_i x_0 Q_m x_m \overleftarrow{P} x_i^+ b \overleftarrow{P} x_m^+ (b^+ \overrightarrow{P} u) & \text{if } x_i^+ b \in E(G). \end{cases}$$

Then P' is contained in $\mathcal{P}$ and longer than P, a contradiction. $\square$

Let $K' = K - \{x_m^+\}$.

Claim 6 *$N_M(K') \cap V(M-K') \subset X \cup \{x_m^+\}$. Furthermore, if $|V(K)| = 1$ or $x_m \in N_M(K')$, then $N_M(K) \cap V(M-K) \subset X$.*

Proof. Clearly, $N_M(K') \cap V(M-K'-P) = \emptyset$.

Suppose that there exists $y \in V(P-K) - X$ such that $yw \in E(G)$ for some $w \in V(K')$. Note that $w^- \in V(K)$. Then $X^+ \cup \{x_0, y^+\}$ is not an independent set, and hence by Claim 2 (iv), $y^+z \in E(G)$ for some $z \in X^+ \cup \{x_0\}$. Therefore let

$$P' = \begin{cases} x_1 \overrightarrow{P} yw \overrightarrow{P} uw^- \overleftarrow{P} y^+ x_0 & \text{if } z = x_0, \\ x_1 \overrightarrow{P} x_i Q_i x_0 Q_m x_m \overleftarrow{P} y^+ x_i^+ \overrightarrow{P} yw \overrightarrow{P} uw^- \overleftarrow{P} x_m^+ & \text{if } z = x_i^+ \in V(x_1 \overrightarrow{P} y), \\ x_1 \overrightarrow{P} yw \overrightarrow{P} uw^- \overleftarrow{P} x_i^+ y^+ \overrightarrow{P} x_i Q_i x_0 & \text{if } z = x_i^+ \in V(y^+ \overrightarrow{P} x_m). \end{cases}$$

Then $P' \in \mathcal{P}$ and P' is longer than P, a contradiction.

In the case $|V(K)| = 1$ or $x_m \in N_M(K')$, we can show $N_M(K) \cap V(M-K) \subset X$ in a similar way. $\square$

Claim 7 *We may assume that $x_1 u \in E(M)$.*

Proof. First, suppose that $|V(K)| = 1$ or $x_m \in N_M(K')$. Since M is m-connected, K has at least m neighbors in $V(M-K)$. By Claim 6, $N_M(K) \cap V(M-K) \subset X$ and $|X| = m$, and hence we have $x_1 \in N_M(w)$ for some $w \in V(K)$. In this case, since K is complete, we can change the path P so that x_1 and w are end-vertices.

Thus, we may assume that $|V(K)| \geq 2$ and $x_m \notin N_M(K')$. In this case, since M is m-connected, K' has at least m neighbors in $V(M-K) \cup \{x_m^+\}$. By Claim 6, $N_M(K') \cap V(M-K) \subset (X - \{x_m\}) \cup \{x_m^+\}$. If $m \geq 2$, then we have $x_1 \in N_M(w)$ for some $w \in V(K)$, and again, we can change the path P so that x_1 and w are end-vertices. Therefore, we may assume that $m = 1$. Then we have $\alpha(M) \leq 2$, and hence $M - P$ is complete. Since x_0

is an arbitrary vertex in $M-P$, we have $N_M(M-P)\cap V(P)\subset\{x_1\}$. If there exists $x'\in V(M-P)$ such that $x_1\notin N_M(x')$, then $\{x',x_1,w\}$ is an independent set of order 3 for any $w\in V(K)-\{x_1^+\}$, a contradiction. Thus, we have $(M-P)\cup\{x_1\}$ is complete and this implies that $M\in\mathcal{G}$. Moreover, if $\{x_1^+\}\neq S-\{x_1\}$, then we can easily find a path which is contained in $\mathcal{P}$ and longer than P. Therefore $S=\{x_1,x_1^+\}$. This implies that (M,S) is an exception pair. $\square$

Since $|S|=m+1$ and $|X|=m$, there exists $y\in S-X$. Then $X^+\cup\{x_0,y^+\}$ is a set of order $m+2$. By Claim 2 (iv), there exists $z\in X^+\cup\{x_0\}$ such that $y^+z\in E(G)$. Let

$$P'=\begin{cases} y\overleftarrow{P}x_1u\overleftarrow{P}y^+x_0 & \text{if } z=x_0,\\ y\overleftarrow{P}x_i^+y^+\overrightarrow{P}ux_1\overrightarrow{P}x_iQ_ix_0 & \text{if } z=x_i^+\in X\cap V(x_1\overrightarrow{P}y),\\ y\overleftarrow{P}x_1u\overleftarrow{P}x_i^+y^+\overrightarrow{P}x_iQ_ix_0 & \text{if } z=x_i^+\in X\cap V(y^+\overrightarrow{P}x_m).\end{cases}$$

Then P' is contained in $\mathcal{P}$ and longer than P, a contradiction. This completes the proof of Theorem 9. $\square$

Proof of Theorem 4.

Let G be a graph and f be a mapping from $V(G)$ to positive integers satisfying the assumption of Theorem 4. Let $S_1=S_1(f)$ and $s_1=s_1(f)$. If $s_1=n$ or $s_1=n+1$, then we have $\alpha(G)=1$, and obviously the statement holds. Therefore we may assume that $n-s_1\geq 1$. Let $m=n-s_1$, $M=G-S_1$ and $g=f|_{V(M)}$. Then M is m-connected, $s_1(g)=0$ and $s_2(g)\leq m+1$.

Moreover,

$$\begin{aligned}\alpha(M) &\leq \alpha(G)\\ &\leq \min_R \Big\{ \sum_{x\in R}(f(x)-1) : R\subset V(G), |R|=n\Big\}+1\\ &= \min_{R'} \Big\{ \sum_{x\in R'}(g(x)-1) : R'\subset V(M), |R'|=m\Big\}+1.\end{aligned}$$

Thus, M satisfies the assumption of Theorem 6.

Case 1. There exists a spanning g-tree T such that $|A_T(g)| \leq m$.

Let $Y = V(M) - A_T(g)$ and

$$Y' = \{y_i : y\in Y, 1\leq i\leq g(y)-d_T(y)\}.$$

We construct a bipartite graph G' as follows;

$$\begin{aligned} V(G') &= S_1 \cup Y'\\ \text{and } E(G') &= \{xy_i : xy\in E(G)\}.\end{aligned}$$

Then by the assumption "$\sum_{x\in V(G)} f(x) \geq 2(|V(G)|-1)$", we obtain

$$\begin{aligned} |Y'| &= \sum_{y\in Y}(g(y)-d_T(y))\\ &= \sum_{y\in V(M)} g(y) - \sum_{y\in V(M)} d_T(y)\\ &= \sum_{y\in V(M)} g(y) - 2(|V(M)|-1)\\ &= \sum_{x\in V(G)} f(x) - s_1 - 2(|V(G)|-s_1-1)\\ &= \sum_{x\in V(G)} f(x) - 2(|V(G)|-1)+s_1\\ &\geq s_1.\end{aligned}$$

Since $G-A_T(g)$ is s_1-connected, for any $\tilde{S}\subset S_1$, $|N_{G-A_T(g)}(\tilde{S})| \geq s_1$, and hence $|N_{G'}(\tilde{S})| \geq |N_{G-A_T(g)}(\tilde{S})| - |S_1-\tilde{S}| \geq |\tilde{S}|$. Therefore, by Hall's Theorem, there exists a matching E' between S_1 and Y' which covers S_1. Let

$E = \{xy : xy_i \in E'$ for some i with $1 \leq i \leq f(y) - d_T(y)\}$. Then $T + E$ is a desired spanning f-tree. $\square$

Case 2. M has no spanning g-tree such that the number of g-saturated vertices is at most m.

In this case, $m = 1$, $|S_2(g)| = s_2(g) = 2$ and $(M, S_2(g))$ is an exception pair. Then we have $s_1 = n - 1$, and $\alpha(G) \leq 2$. If $\alpha(G) = 1$, then obviously the statement is true. Thus, we may assume that $\alpha(G) = 2$. Let $U = S_2(g) = \{u, u'\}$.

Let H_1 and H_2 be components of $M - U$ and $Y_i = V(H_i)$. Clearly, there exists a Hamilton path T in M. Note that $A_T(g) = U$. Let

$$Y_l' = \{y_i : y \in Y_l, 1 \leq i \leq g(y) - d_T(y)\}$$

for $l = 1, 2.$ and let $Y' = Y_1' \cup Y_2'$. Note that $Y_1' \neq \emptyset$ and $Y_2' \neq \emptyset$. Again, we construct a bipartite graph G' as follows;

$$\begin{aligned} V(G') &= S_1 \cup Y' \\ \text{and } E(G') &= \{xy_i : xy \in E(G)\}. \end{aligned}$$

By the same argument as Case 1, we have $|Y'| \geq s_1 = |S_1|$. Again, we will show that there exists a matching E' between S_1 and Y' which covers S_1. Assume that there exists no such matching. Then by Hall's Theorem, there exists $\tilde{S} \subset S_1$ such that $|N_{G'}(\tilde{S})| < |\tilde{S}|$.

Claim 8 *$Y_1' \subset N_{G'}(\tilde{S})$ or $Y_2' \subset N_{G'}(\tilde{S})$.*

Proof. Suppose that there exist $y_i \in Y_1'$ and $z_j \in Y_2'$ such that $y_i, z_j \notin N_{G'}(\tilde{S})$. Then by the definition of G', for any $x \in \tilde{S}$, $\{x, y, z\}$ is an independent set of order 3, which contradicts $\alpha(G) = 2$. $\square$

By the symmetry, suppose that $Y_1' \subset N_{G'}(\tilde{S})$ holds. Let $Z_2' = N_{G'}(\tilde{S}) \cap Y_2'$ and $Z_2 = \{y : y_i \in Z_2'\}$. Since $|Y_1'| + |Z_2'| = |N_{G'}(\tilde{S})| < |\tilde{S}|$, we have

$$|Z_2'| \leq |\tilde{S}| - |Y_1'| - 1 \leq |\tilde{S}| - 2,$$

and hence

$$\begin{aligned} |Z_2'| + |S_1 - \tilde{S}| + |U| &\leq |\tilde{S}| - 2 + |S_1 - \tilde{S}| + 2 \\ &= |S_1| \\ &= n - 1. \end{aligned} \tag{5}$$

On the othe hand, since $|Y'| \geq |S_1|$ and $|Y_1'| + |Z_2'| < |\tilde{S}|$, we obtain

$$|Y_2' - Z_2'| = (|Y'| - |Y_1'|) - |Z_2'| > |S_1| - |\tilde{S}| \geq 0.$$

Thus, $Y_2' - Z_2' \neq \emptyset$ and hence $Y_2 - Z_2 \neq \emptyset$. This implies that in the graph G, $Z_2 \cup (S_1 - \tilde{S}) \cup U$ separates Y_1 from $Y_2 - Z_2$, and hence

$$\begin{aligned} |Z_2'| + |S_1 - \tilde{S}| + |U| &\geq |Z_2| + |S_1 - \tilde{S}| + |U| \\ &\geq n, \end{aligned}$$

because G is n-connected. This contradicts the inequality (5).

Therefore there exists a matching E' between S_1 and Y' which covers S_1. Let $E = \{xy : xy_i \in E' \text{ for some } i \text{ with } 1 \leq i \leq g(y) - d_T(y)\}$. Again, $T + E$ is a desired spanning f-tree. $\square$

3 Conclusion

In this paper, we have given a partial solution to Conjecture 3. The remaining part of this conjecture can be restated as follows:

Conjecture 11 *Let n be an integer, G be an n-connected graph and f be a mapping from $V(G)$ to the positive integers. If $\sum_{x \in V(G)} f(x) \geq 2(|V(G)|-1)$ and $\alpha(G) \leq n - s_1(f) + 1$, then there exists a spanning f-tree.*

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