Dominating cycles in triangle-free graphs

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Abstract

A cycle C in a graph G is said to be dominating if $E(G-C)=\emptyset$. Enomoto et al. showed that if G is a 2-connected triangle-free graph with $\alpha(G) \le 2\kappa(G)-2$, then every longest cycle is dominating. But it is unknown whether the condition on the independence number is sharp. In this paper, we show that if G is a 2-connected triangle-free graph with $\alpha(G) \le 2\kappa(G)-1$, then G has a longest cycle which is dominating. This condition is best possible.

1 Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. We denote the degree of a vertex x by $d_G(x)$. Let $\alpha(G)$ and $\kappa(G)$ be the independence number and the connectivity of a graph G, respectively.

For a graph G, we define $\sigma_3(G) := \min\{d_G(x) + d_G(y) + d_G(z) : \{x, y, z\}$ is an independent set in $G\}$ if $\alpha(G) \geq 3$, and $\sigma_3(G) := +\infty$ if $\alpha(G) < 3$. A cycle C is *dominating* in G if $E(G - C) = \emptyset$ holds. Bondy showed the following theorem.

Theorem 1 (Bondy [2]) *Let G be a 2-connected graph. If $\sigma_3(G) \geq |V(G)| + 2$, then any longest cycle of G is dominating.*

The degree condition of Theorem 1 is best possible in a sense. Let $G_1 = K_m + (m + 1)K_2$ with $m \geq 2$. Then G_1 satisfies $\sigma_3(G_1) = 3(m + 1) = |V(G_1)| + 1$ and has no dominating cycle. Aung showed that if we restrict ourselves to triangle-free graphs, then we can relax the lower bound. Let $\delta(G)$ be the minimum degree of G.

Theorem 2 (Aung [1]) *Let G be a 2-connected triangle-free graph. If $\delta(G) \geq \frac{1}{6}(|V(G)| + 6)$, then there exists a longest cycle of G which is dominating.*

The following theorem is a corollary of a theorem in [3]. This is a generalization of Theorem 2.

Theorem 3 (Broersma et al. [3]) *Let G be a 2-connected triangle-free graph. If $\sigma_3(G) \geq \frac{1}{2}(|V(G)| + 5)$, then there exists a longest cycle of G which is dominating.*

On the other hand, Chvátal and Erdős considered a condition concerning the independence number and connectivity. This is called a *Chvátal-Erdős type* condition.

Theorem 4 (Chvátal and Erdős [4]) *If a graph G is 2-connected and $\alpha(G) \leq \kappa(G)$, then G is hamiltonian.*

Enomoto et al. considered the relationship between a Chvátal-Erdős type condition and the existence of dominating cycles.

Theorem 5 (Enomoto et al. [5]) *Let G be a 2-connected triangle-free graph. If $\alpha(G) \leq 2\kappa(G) - 2$, then any longest cycle of G is dominating.*

They also showed that the condition cannot be replaced with $\alpha(G) \leq 2\kappa(G)$ by giving a triangle-free graph G with $\alpha(G) = 2\kappa(G)$ that does not have a dominating cycle.

In this paper, we improve the condition of Theorem 5 for the existence of dominating cycles. We prove the following result.

Theorem 6 *Let G be a 2-connected triangle-free graph. If $\alpha(G) \leq 2\kappa(G) - 1$, then G has a longest cycle which is dominating.*

We do not know whether any longest cycle is dominating under the condition of Theorem 6.

2 Proof of Theorem 6

For graph-theoretic terminology not explained in this paper, we refer the reader to [6]. For $U \subset V(G)$, we define $N_G(U)$ by $N_G(U) := \bigcup_{u \in U} N_G(u)$. For a subgraph H of G, we write $N_H(u)$, $N_H(U)$ and $d_H(u)$ instead of $N_G(u) \cap V(H)$, $N_G(U) \cap V(H)$ and $|N_H(u)|$, respectively. We sometimes write $N_G(H)$ instead of $N_G(V(H))$.

Let C be a cycle in G. We give an orientation to C and write the oriented cycle C by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote an xy-path on $\overrightarrow{C}$ by $x\overrightarrow{C}y$, and write the reverse sequence of $x\overrightarrow{C}y$ by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. For $X \subset V(C)$, we define $X^{+h} := \{x^{+h} : x \in X\}$ and $X^{-h} := \{x^{-h} : x \in X\}$. We often write x^+, x^-, X^+ and X^- for x^{+1}, x^{-1}, X^{+1} and X^{-1}, respectively.

A path joining u and v is called a *uv-path*. For a subgraph H, a uv-path P is called an *H-path* if $V(P) \cap V(H) = \{u, v\}$ and $E(P) \cap E(H) = \emptyset$. A graph G is said to be *k-path-connected* if for every pair of distinct vertices

u and v, there exists a uv-path of length at least k. The following lemma is used in the proof of Claim 2.

Lemma 1 ([5, Lemma 3]) *Let G be a 2-connected triangle-free graph and let $x_0 \in V(G)$. If $d_G(x) \geq 3$ for each $x \in V(G) - \{x_0\}$, then G is 4-path-connected.* $\Box$

Proof of Theorem 6. Let C be a cycle in G such that

(C1) C is a longest cycle in G, and

(C2) $|E(G - C)|$ is as small as possible, subject to (C1).

If $E(G - C) = \emptyset$, then there is nothing to prove. Therefore we may assume $E(G-C) \neq \emptyset$, and let H be a component of $G-C$ with $|V(H)| \geq 2$.

By (C1), the following fact holds.

Fact 1 (i) $N_C(H) \cap N_C(H)^+ = \emptyset$.

(ii) *There exists no C-path joining two vertices of $N_C(H)^+$ (or $N_C(H)^-$).*

(iii) *Let $u_i \in V(H)$ and $x_i \in N_C(u_i)$ for $i = 1, 2$. If $x_1 \neq x_2$ and there exists a u_1u_2-path in H of length at least k, then $x_1^{+k} \neq x_2$ and $x_1^{+l}x_2^{+m} \notin E(G)$ for any positive integers l, m with $l + m = k + 2$.*

$\Box$

By Fact 1, we obtain the following fact.

Fact 2 *Let S be an independent set in H. Then $S \cup N_C(H)^+$ is independent. In particular, for every $v \in V(H)$, $N_H(v) \cup N_C(H)^+$ is an independent set.*
$\Box$

We will show several claims concerning the structure of H.

Claim 1 *For every $u \in V(H)$, $N_C(u) \neq \emptyset$.*

Proof. By Fact 2, for every $u \in V(H)$, $N_H(u) \cup N_C(H)^+$ is an independent set. Since $|N_C(H)^+| = |N_C(H)| \geq \kappa(G)$, we have

$$\begin{aligned} 2\kappa(G) - 1 \ \geq \ \alpha(G) \ &\geq \ |N_H(u) \cup N_C(H)^+| \\ &\geq \ d_H(u) + \kappa(G). \end{aligned}$$

This implies $d_H(u) \leq \kappa(G) - 1$. Therefore, we obtain $d_C(u) = d_G(u) - d_H(u) \geq \kappa(G) - (\kappa(G) - 1) = 1$. □

Claim 2 $\delta(H) \leq 2$.

Proof. We use a similar argument as the proof of Theorem 1 in [5].

Suppose $\delta(H) \geq 3$.

Case 1 : H is not 2-connected.

Let B be an end block of H, $c_B \in V(B)$ be the cut-vertex of H and B' be an end block of H other than B. Since $d_B(x) \geq 3$ for every $x \in V(B) - \{c_B\}$, by Lemma 1, B is 4-path-connected. Let $T := N_C(B - \{c_B\})$,

$$T_0 := \{x \in T : N_{B-\{c_B\}}(x) = N_{B-\{c_B\}}(x_0) = \{u\} \text{ for some } x_0 \in T - \{x\} \text{ and } u \in V(B) - \{c_B\}\},$$

and $T_1 := T - T_0$. And let S_B ($S_{B'}$) be a maximum independent set in B (B', respectively). Since $\delta(H) \geq 3$ and G is triangle-free, we have $|S_B|, |S_{B'}| \geq 3$. Moreover $|V(B)| \geq 4$. Let $S := ((S_B \cup S_{B'}) - \{c_B\}) \cup T^+ \cup T_1^{+3}$.

Claim 2.1 *S is an independent set of order at least $2|T_1| + |T_0| + 4$.*

Proof. By Fact 2, $((S_B \cup S_{B'}) - \{c_B\}) \cup T^+$ is an independent set.

For every $x_1, x_2 \in T_1$ ($x_1 \neq x_2$), by the definition of T_1, there exist vertices $u \in N_{B-\{c_B\}}(x_1)$ and $v \in N_{B-\{c_B\}}(x_2)$ such that $u \neq v$. Since B is 4-path-connected, there exists a uv-path P in B with length at least 4. Therefore by Fact 1 (iii), we have $x_1^{+3}x_2^{+3} \notin E(G)$, and hence T_1^{+3} is independent.

By the similar argument, we can show that any vertex in T_1^{+3} is not adjacent to any vertex in $((S_B \cup S_{B'}) - \{c_B\}) \cup T^+$, and hence S is an independent set.

Moreover we have

$$\begin{aligned}|S| &\geq (|S_B|-1)+(|S_{B'}|-1)+|T^+|+|T_1^{+3}| \\ &\geq 2+2+|T|+|T_1| \\ &= 2|T_1|+|T_0|+4. \qquad \square\end{aligned}$$

Case 1-1 : $N_{B-\{c_B\}}(T_0) \neq V(B)-\{c_B\}$.

Then $T_1 \cup N_{B-\{c_B\}}(T_0) \cup \{c_B\}$ is a separating set, and hence

$$|T_1 \cup N_{B-\{c_B\}}(T_0) \cup \{c_B\}| = |T_1| + |N_{B-\{c_B\}}(T_0)| + 1 \geq \kappa(G).$$

By the definition of T_0, $|N_{B-\{c_B\}}(T_0)| \leq \frac{1}{2}|T_0|$, and therefore $|T_1|+\frac{1}{2}|T_0|+1 \geq \kappa(G)$. This implies $2|T_1|+|T_0|+2 \geq 2\kappa(G)$. Thus, by Claim 2.1 we obtain

$$2\kappa(G)-1 \geq \alpha(G) \geq |S| \geq 2|T_1|+|T_0|+4 \geq 2\kappa(G)+2,$$

a contradiction.

Case 1-2 : $N_{B-\{c_B\}}(T_0) = V(B)-\{c_B\}$.

Choose $u \in V(B)-\{c_B\}$ so that $|N_C(u) \cap T_0|$ is as small as possible. Let $d_0 := |N_C(u) \cap T_0|$, $d_1 := |N_C(u) \cap T_1|$ and $b := |V(B)|$. Then $|T_1| \geq d_1$ and $b \geq 4$. By the definition of T_0, we have $d_0 \geq 2$. Since $N_C(u) \cup (V(B)-\{u\})$ is a separating set,

$$|N_C(u) \cup (V(B)-\{u\})| = d_0+d_1+b-1 \geq \kappa(G). \tag{1}$$

By the definition of T_0 and the choice of u, $|T_0| \geq d_0(b-1)$. Hence by Claim 2.1, we obtain

$$|S| \geq 2|T_1|+|T_0|+4 \geq 2d_1+d_0(b-1)+4. \tag{2}$$

By (1) and (2), we have

$$2(d_0+d_1+b-1)-1 \geq 2\kappa(G)-1 \geq \alpha(G) \geq 2d_1+d_0(b-1)+4.$$

This implies that $(d_0-2)(b-3)+1 \le 0$, contradicting that $d_0 \ge 2$ and $b \ge 4$.

Case 2 : H is 2-connected.

In this case, we use a similar argument as in Case 1. By Lemma 1, H is 4-path-connected. Let $T := N_C(H)$, $T_0 := \{x \in T : N_H(x) = N_H(x_0) = \{u\}$ for some $x_0 \in T - \{x\}$, and $u \in V(H)\}$, $T_1 := T - T_0$, and let S_H be a maximum independent set in H. Then $|S_H| \ge 3$ and $|V(H)| \ge 4$.

Let $S := S_H \cup T^+ \cup T_1^{+3}$. Then by the same argument in the proof of Claim 2.1, S is an independent set with $|S| = |S_H| + |T^+| + |T_1^{+3}| \ge 3 + |T| + |T_1| = 2|T_1| + |T_0| + 3$.

Case 2-1 : $N_H(T_0) \ne V(H)$.

Since $T_1 \cup N_H(T_0)$ is a separating set and $|N_H(T_0)| \le \frac{1}{2}|T_0|$, we have $|T_1| + \frac{1}{2}|T_0| \ge \kappa(G)$. Therefore we have

$$2\kappa(G) - 1 \ge \alpha(G) \ge |S| \ge 2|T_1| + |T_0| + 3 \ge 2\kappa(G) + 3,$$

a contradiction.

Case 2-2 : $N_H(T_0) = V(H)$.

By the same way as in Case *1-2*, choose $u \in V(H)$ so that $|N_C(u) \cap T_0|$ is as small as possible, and let $d_0 := |N_C(u) \cap T_0|$ and $d_1 := |N_C(u) \cap T_1|$. Clearly $|V(H)| \ge 4$. Because $N_C(u) \cup (V(H) - \{u\})$ is a separating set, we have $d_0 + d_1 + |V(H)| - 1 \ge \kappa(G)$.

On the other hand, since $|T_0| \ge d_0|V(H)|$, $|S| \ge 2d_1 + d_0|V(H)| + 3$. Therefore $2(d_0 + d_1 + |V(H)| - 1) - 1 \ge 2d_1 + d_0|V(H)| + 3$, and then $(d_0 - 2)(|V(H)| - 2) + 2 \le 0$. This contradicts that $d_0 \ge 2$ and $|V(H)| \ge 4$.

This complete the proof of Claim 2. □

Claim 3 $\delta(H) = 1$.

Proof. Assume $\delta(H) = 2$. Let u be a vertex in H with $d_H(u) = 2$, and let $N_H(u) = \{v_1, v_2\}$. Without loss of generality, we may assume that $|N_C(v_1)| \le |N_C(v_2)|$. Let $S := N_C(\{u, v_1\})^+ \cup N_H(v_1)$. By Fact 2, S is an independent set. Since $N_C(u) \cap N_C(v_1) = \emptyset$, we have

$$\begin{aligned} |S| &= |N_C(u)^+| + |N_C(v_1)^+| + |N_H(v_1)| \\ &= d_C(u) + d_G(v_1) \\ &\ge (\kappa(G) - 2) + \kappa(G) = 2\kappa(G) - 2. \end{aligned} \tag{3}$$

Since $\delta(H) = 2$, there exists $w_2 \in N_H(v_2) - \{u\}$. By Claim 1, there exists $x_2 \in N_C(w_2)$.

Claim 3.1 *(i)* $S \cup \{x_2^{+3}\}$ *is an independent set of order* $2\kappa(G) - 1$.

(ii) $d_C(u) = \kappa(G) - 2$.

Proof. There exist a uw_2-path and a v_1w_2-path in H of length at least 2. By Fact 1 (iii), $x_2^{+2} \notin N_C(\{u, v_1\})$, and hence $x_2^{+3} \notin S$. Again, by Fact 1 (iii), $x^+x_2^{+3} \notin E(G)$ for any $x \in N_C(\{u, v_1\}) - \{x_2\}$. Since G is triangle-free, $x_2^+x_2^{+3} \notin E(G)$. Hence $N_C(\{u, v_1\})^+ \cup \{x_2^{+3}\}$ is independent.

For any $w \in N_H(v_1) - \{w_2\}$, there exists a ww_2-path in H of length at least 2. By Fact 1 (iii), $x_2^{+3} \notin N_C(w)$. Therefore $(N_H(v_1) - \{w_2\}) \cup \{x_2^{+3}\}$ is independent. Suppose $x_2^{+3} \in N_C(w_2)$. Then by Fact 1 (iii), $x_2, x_2^{+3} \notin N_C(\{u, v_1\})$. Therefore by Fact 2, $S \cup \{x_2^+, x_2^{+4}\}$ is an independent set of order $2\kappa(G)$, a contradiction. Thus, we have $x_2^{+3} \notin N_C(w_2)$. Therefore we have $S \cup \{x_2^{+3}\}$ is an independent set. Since $\alpha(G) \le 2\kappa(G) - 1$, inequality (3) implies that $|S \cup \{x_2^{+3}\}| = 2\kappa(G) - 1$ and $d_C(u) = \kappa(G) - 2$. □

Claim 3.2 $x_2 \in N_C(\{u, v_1\})$.

Proof. Suppose not. Then $x_2^+ \notin S$. Since G is triangle-free, $x_2^+x_2^{+3} \notin E(G)$. By Claim 3.1 and Facts 1 (i) and (ii), $S \cup \{x_2^+, x_2^{+3}\}$ is an independent set of order $2\kappa(G)$, a contradiction. □

Claim 3.3 *$w_2 \in N_H(v_1)$ or $N_H(w_2) \cap N_H(v_1) \neq \emptyset$.*

Proof. Suppose that $w_2 \notin N_H(v_1)$ and $N_H(w_2) \cap N_H(v_1) = \emptyset$. Then $N_H(v_1) \cup \{w_2\}$ is independent. Since there exist a w_2u-path and a w_2v_1-path of length at least 2, by Claim 3.2, $S \cup \{w_2, x_2^{+3}\}$ is an independent set. This contradicts that $\alpha(G) \leq 2\kappa(G) - 1$. □

Claim 3.4 *$N_C(v_1) = N_C(v_2)$.*

Proof. By Claim 3.3, there exist a v_2u-path and a v_2v_1-path in H of length at least 2. Hence $S \cup \{x_2^{+3}\} \cup N_C(v_2)^+$ is an independent set. Since $\alpha(G) \leq 2\kappa(G) - 1$ and G is triangle-free, $N_C(v_2) \subset N_C(v_1)$. Thus, since $|N_C(v_1)| \leq |N_C(v_2)|$, we have $N_C(v_1) = N_C(v_2)$. □

Claim 3.5 *$N_C(H) = N_C(\{u, v_1\})$.*

Proof. Suppose that $N_C(H) - N_C(\{u, v_1\}) \neq \emptyset$. Let $w \in V(H)$ such that $N_C(w) - N_C(\{u, v_1\}) \neq \emptyset$. By Claim 3.4, we have $w \notin \{u, v_1, v_2\}$, and hence there exist a wu-path and a wv_1-path or a wv_2-path in H of length at least 2. Then $|S \cup \{x_2^{+3}\} \cup N_C(w)^+| \geq |S| + 2 = 2\kappa(G)$, a contradiction. □

Claim 3.6 *$|N_C(v_1)| = 1$.*

Proof. Assume that $|N_C(v_1)| \geq 2$. Let $x_0, x_1 \in N_C(v_1)$ with $x_0 \neq x_1$. By Claim 3.4, $x_0, x_1 \in N_C(v_2)$. Since G is triangle-free, we have$x_2 \neq x_0, x_1$. By Claim 3.3, there exists a uv_1-path in H of length at least 2. Hence $S \cup \{x_i^{+3}\}$ is an independent set for $i = 0, 1$. Since G is triangle-free, there exist $i, j \in \{0, 1, 2\}$ such that $x_i^{+3}x_j^{+3} \notin E(G)$. Then $S \cup \{x_i^{+3}, x_j^{+3}\}$ is an independent set of order $2\kappa(G)$, a contradiction. □

By Claims 3.1 (ii), 3.5 and 3.6, $|N_C(H)| = |N_C(u)| + |N_C(v_1)| = (\kappa(G) - 2) + 1 = \kappa(G) - 1$. This contradicts the connectivity of G, and completes the proof of Claim 3. □

Claim 4 *H is a star.*

Proof. Suppose that H is not a star. By Claim 3, there exists a vertex $u \in V(H)$ with $d_H(u) = 1$. Since H is not a star, there exists a path uvw_1w_2 in H of length 3. Note that $w_2 \notin N_H(v)$ since G is triangle-free. Let $S := N_C(\{u, v\})^+ \cup N_H(v)$. Since

$$\begin{aligned} |S| &= d_C(u) + d_C(v) + d_H(v) \\ &\geq \kappa(G) - 1 + \kappa(G) \\ &= 2\kappa(G) - 1, \end{aligned}$$

it follows from Fact 2 that S is a maximum independent set.

By Claim 1, there exists $x_2 \in N_C(w_2)$.

We show that $(S - \{w_1\}) \cup \{x_2^{+3}\}$ is also a maximum independent set. There exist a w_2u-path and a w_2v-path in H of length at least 2. By Fact 1 (iii), $x_2^{+2} \notin N_C(\{u, v\})$ and $x^+x_2^{+3} \notin E(G)$ for any $x \in N_C(\{u, v\}) - \{x_2\}$. Since G is triangle-free, $x_2^+x_2^{+3} \notin E(G)$. Thus, $x_2^{+3} \notin N_C(\{u, v\})^+$ and $N_C(\{u, v\})^+ \cup \{x_2^{+3}\}$ is independent. For any $w \in N_H(v) - \{w_1\}$, there exist a ww_2-path in H of length at least 2. By Fact 1, $(N_H(v) - \{w_1\}) \cup \{x_2^{+3}\}$ is independent. Therefore $(S - \{w_1\}) \cup \{x_2^{+3}\}$ is a maximum independent set.

Suppose that $N_H(w_2) \cap N_H(v) \neq \{w_1\}$. Then there exists a w_1w_2-path in H of length at least 2. Thus, $S \cup \{x_2^{+3}\}$ is a independent set of order $2\kappa(G)$, a contradiction.

Therefore we may assume that $N_H(w_2) \cap N_H(v) = \{w_1\}$. By Fact 2, $S \cup \{x_2^+\}$ is independent. Since S is a maximum independent set, $x_2 \in N_C(\{u, v\})$. Since there exist a w_2u-path and a w_2v-path in H of length at least 2, by Fact 1 (iii) we have $x_2^{+3} \notin N_C(w_2)$. Therefore $(S - \{w_1\}) \cup \{w_2, x_2^{+3}\}$ is an independent set of order $2\kappa(G)$, a contradiction. This completes the proof of Claim 4. □

Let v be the center vertex of H and $X := N_C(H)^+$. By Fact 2, $N_H(v) \cup X$

is independent. For every $u \in N_H(v)$, we obtain $|X| = |N_C(H)| \geq d_C(v) + d_C(u) \geq d_C(v) + (d_G(u) - 1)$. Therefore $|N_H(v)| + |X| \geq d_G(v) + \kappa(G) - 1 \geq 2\kappa(G) - 1$.

Let $X_0 := \{x \in X : N_{G-C}(x) = \emptyset\}$ and $X_1 := X - X_0 = \{x \in X : N_{G-C}(x) \neq \emptyset\}$. By (C2), we obtain the following fact.

Fact 3 (i) *$N_C(H) \cap X_0^+ = \emptyset$.*

(ii) *There exists no C-path joining a vertex of X and a vertex of X_0^+.*

For each $x \in X_1$, we choose an arbitrary vertex x^* of $N_{G-C}(x)$, and let $Y^* := \{x^* : x \in Y\}$ for $Y \subset X_1$. By Fact 1 (ii), for any $x_1, x_2 \in X_1$ with $x_1 \neq x_2$, we have $x_1^* \neq x_2^*$ and $x_1^* x_2^* \notin E(G)$. Moreover, for any $x_1 \in X_1$ and $x_2 \in X$ with $x_1 \neq x_2$, we have $x_1^* x_2 \notin E(G)$. Therefore for every $Y_1 \subset X_1$, $Y_1^* \cup (X - Y_1)$ is independent and $|Y_1^*| = |Y_1|$. By Fact 1 (i), $N_H(v) \cup X_1^*$ is an independent set.

By Fact 3, $N_G(x_0^+) \cap (N_H(v) \cup (X - \{x_0\})) = \emptyset$ for every $x_0 \in X_0$. Moreover we have $N_G(x_0^+) \cap (X_1^* \cup (X_0 - \{x_0\})) = \emptyset$.

We consider two cases.

Case 1 : There exists $x \in X$ such that $x^+, x^{+2} \notin N_C(H)$.

Let $x \in X$ such that $x^+, x^{+2} \notin N_C(H)$. We partition X_i into Y_i and Z_i for $i = 0, 1$ so that $Y_i := X_i \cap N_G(x^{+2})$, and $Z_i := X_i - N_G(x^{+2})$. By the triangle-free condition, $(Y_0^+ \cup Y_1^*) \cap N_G(x^{+2}) = \emptyset$. Since $|X_i| = |Y_i| + |Z_i|$ for $i = 0, 1$, we have

$$\begin{aligned}
&|N_H(v) \cup Y_0^+ \cup Z_0 \cup Y_1^* \cup Z_1 \cup \{x^{+2}\}| \\
&= |N_H(v)| + |X_0| + |X_1| + 1 \\
&= |N_H(v)| + |X| + 1 \\
&\geq 2\kappa(G).
\end{aligned}$$

Therefore $N_H(v) \cup Y_0^+ \cup Z_0 \cup Y_1^* \cup Z_1 \cup \{x^{+2}\}$ is not an independent set, and hence there exist $x_1, x_2 \in Y_0$ such that $x_1^+ x_2^+ \in E(G)$.

Claim 5 $N_H(v) \cup X \cup \{x^{+3}\}$ *is an independent set.*

Proof. Without loss of generality, we may assume that $x^{+3} \in V(x_2^{+3}\overrightarrow{C}x_1^-)$.

First, we show that $N_G(x^{+3}) \cap X = \emptyset$. Suppose that there exists $x_3 \in N_G(x^{+3}) \cap X$. Since G is triangle-free, we have $x_3 \neq x_1, x_2$. Let Q be a C-path joining x_3^- and x_2^-. We define a cycle C' as follows:

$$C' = \begin{cases} x^{+3}\overrightarrow{C}x_3^- Q x_2^- \overleftarrow{C} x_1^+ x_2^+ \overrightarrow{C} x^{+2} x_1 \overleftarrow{C} x_3 x^{+3} & \text{if } x_3 \in V(x^{+3}\overrightarrow{C}x_1^-), \\ x^{+3}\overrightarrow{C}x_1 x^{+2} \overleftarrow{C} x_2^+ x_1^+ \overrightarrow{C} x_3^- Q x_2^- \overleftarrow{C} x_3 x^{+3} & \text{if } x_3 \in V(x_1^+\overrightarrow{C}x_2^-), \\ x^{+3}\overrightarrow{C}x_2^- Q x_3^- \overleftarrow{C} x_2 x^{+2} \overleftarrow{C} x_3 x^{+3} & \text{if } x_3 \in V(x_2^+\overrightarrow{C}x^{+2}). \end{cases}$$

Note that in each case $V(C') \supset V(C) - \{x_2\}$ and C' passes a vertex in H. Since $x_2 \in Y_0 \subset X_0$, $N_{G-C}(x_2) = \emptyset$. Therefore $|V(C')| > |V(C)|$, or $|V(C')| = |V(C)|$ and $|E(G - C')| < |E(G - C)|$, which contradicts (C1) or (C2).

Next, we show that $N_G(x^{+3}) \cap N_H(v) = \emptyset$. Let R be a C-path joining x^{+3} and x_2^-. Then $C' = x^{+3}\overrightarrow{C}x_1 x^{+2}\overleftarrow{C}x_2^+ x_1^+ \overrightarrow{C} x_2^- R x^{+3}$ is a cycle such that $|V(C')| > |V(C)|$ or $|V(C')| = |V(C)|$ and $|E(G-C')| < |E(G-C)|$. Again this contradicts (C1) or (C2). $\square$

Since $x^{+2} \notin N_C(H)$, we have $x^{+3} \notin X$. Therefore $|N_H(v) \cup X \cup \{x^{+3}\}| \geq 2\kappa(G)$, a contradiction.

Case 2 : For every $x \in X$, $x^+ \in N_C(H)$ or $x^{+2} \in N_C(H)$.

By Claim 1, we can choose $w \in N_C(v)$. Note that $N_G(w) \cap N_H(v) = \emptyset$, since G is triangle-free. In this case, we partition X_i into Y_i and Z_i for $i = 0, 1$ so that $Y_i := X_i \cap N_G(w)$ and $Z_i := X_i - N_G(w)$.

By the similar argument as in Case 1, we have $|N_H(v) \cup Y_0^+ \cup Z_0 \cup Y_1^* \cup Z_1 \cup \{w\}| \geq 2\kappa(G)$, and hence there exist $x_1, x_2 \in Y_0$ such that $x_1^+ x_2^+ \in E(G)$.

On the other hand, by Fact 3 (i) $x_1^+, x_2^+ \notin N_C(H)$. Therefore $x_1^{+2}, x_2^{+2} \in N_C(H)$, which implies $x_1^+, x_2^+ \in N_C(H)^-$. Then by Fact 1 (ii), $x_1^+ x_2^+ \notin E(G)$, a contradiction. $\square$

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