On Geometric Plane Graphs induced by Path Decompositions

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A *planar graph* G is a graph embeddable into the euclidean plane $\mathbf{R}^2$. For a planar embedding $f_0 : G \to \mathbf{R}^2$, the image $G_0 = f_0(G)$ is called a *plane graph of* G *corresponding to* f_0 and vice versa. A plane graph $\overline{G_0}$ of G is a *geometric plane graph* if the image of each edge of G is a straight-line segment of $\mathbf{R}^2$. Fáry proved that for every planar graph G there exists a geometric plane graph of G, it is called *Fáry's theorem* [1]. A plane graph G_1 of G is *ambient isotopic* to a plane graph G_2 of G if there exists an orientation preserving homeomorphism $h : \mathbf{R}^2 \to \mathbf{R}^2$ such that the composition $h \circ f_1 = f_2$, where $f_i : G \to \mathbf{R}^2$ is the planar embedding corresponding to G_i for $i = 1, 2$. We can prove *Fáry's ambient isotopy theorem* (in short FAIT): for any planar graph G and any plane graph G_0 of G, there exists a geometric plane graph $\overline{G_0}$ of G such that G_0 is ambient isotopic to $\overline{G_0}$.

A *path decomposition* $\mathcal{P} = \{P_1, \cdots, P_n\}$ of G is a set of n paths of G with positive length such that $\bigcup_{i=1}^{n} E(P_i) = E(G)$ and $E(P_i) \cap E(P_j) = \emptyset$ for $1 \le i < j \le n$, where $E(*)$ is the edge set of $*$. A plane graph $\overline{G_0}$ of G is a *geometric plane graph induced by* $\mathcal{P}$ if the image of each path in $\mathcal{P}$ is a straight-line segment of $\mathbf{R}^2$. We consider *Fáry's ambient isotopy problem induced by a path decomposition*: for any planar graph G, a path decomposition $\mathcal{P}$ of G and a plane graph G_0 of G, how condiditons do $\mathcal{P}$ and G_0 satisfy if and only if there exists a geometric plane graph $\overline{G_0}$ of G induced by $\mathcal{P}$ such that G_0 is ambient isotopic to $\overline{G_0}$? In this abstract we show *FAIT induced by a path decomposition* (in short FAITPD) for certain graphs. In FAITPD a path decomposition $\mathcal{P}$ must be 2-*acyclic*, that is, $P_i \cup P_j$ contains no cycle for $1 \le i < j \le n$. A 2-acyclic path decomposition $\mathcal{P}$ is *transversal in* G_0 if P_i^0 crosses P_j^0 at the vertex $v^0 = P_i^0 \cap P_j^0$ when $P_i \cap P_j \ne \emptyset$ for $1 \le i < j \le n$, where the image of a subgraph H of G is denoted by H^0. We show FAITPD for trees.

Theorem 1. *Let G be a tree, $\mathcal{P}$ a 2-acyclic path decomposition of G and G_0 a plane graph of G. Then $\mathcal{P}$ is transversal in G_0 if and only if there exists a geometric plane graph $\overline{G_0}$ of G induced by $\mathcal{P}$ such that G_0 is ambient isotopic to $\overline{G_0}$.*

A 2-cyclic path decomposition $\mathcal{P}$ is *inessential* if $P_i \cap P_j$ is an endvertex of P_i and P_j when $P_i \cap P_j \ne \emptyset$ for $1 \le i < j \le n$. We shall define a *polygonal path decomposition in* G_0 to describe *FAIT induced by an inessential path decomposition*

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(in short FAITIPD). For any cycle C of G, let $\mathcal{P}_i^C$ be the set of all path-components of the subgraph of G induced by $E(P_i) \cap E(C)$. We set $\mathcal{P}_C = \bigcup_{i=1}^{n} \mathcal{P}_i^C$. A vertex v of C is a *polygonal vertex of* $\mathcal{P}_C$ if v is an endpoint of a path in $\mathcal{P}_C$. We denote the set of all polygonal vertices of $\mathcal{P}_C$ by $V(C;\mathcal{P})$. A polygonal vertex v of $\mathcal{P}_C$ is *not convex in* G_0 if there exists a path P_i in $\mathcal{P}$ such that v is an internal vertex of P_i and $N(v^0) \cap P_i^0 \subset \mathrm{Int}(C^0)$, where $N(v^0)$ is a regular neighborhood of v^0 in $\mathbf{R}^2$ and $\mathrm{Int}(C^0)$ is the union of C^0 and its interior. We denote the set of all polygonal verteices of $\mathcal{P}_C$ which are convex in G_0 by $V_{\mathrm{conv}}(C^0;\mathcal{P})$. A 2-acyclic path decomposition $\mathcal{P}$ is *polygonal in* G_0 if $|V_{\mathrm{conv}}(C^0;\mathcal{P})| \geq 3$ for every cycle of C of G, where $*$ is the number of elements of $*$. We show FAITIPD.

THEOREM 2. *Let G be a planar graph, $\mathcal{P}$ a 2-acyclic inessential path decomposition of G and G_0 a plane graph of G. Then $\mathcal{P}$ is polygonal in G_0 if and only if there exists a geometric plane graph $\overline{G_0}$ of G induced by $\mathcal{P}$ such that G_0 is ambient isotopic to $\overline{G_0}$.*

An ordered triple $(u,\ v,\ w)$ of three distinct vertices of G is a *colinear triple of* G if for any geometric plane graph $\overline{G_0}$, there exists a line ℓ of $\mathbf{R}^2$ such that $\overline{u^0}$, $\overline{v^0}$ and $\overline{w^0}$ are appeared on ℓ in this order. A 2-cyclic path decomposition $\mathcal{P}$ is *colinear in* G_0 if for any colinear triple $(u,\ v,\ w)$ of G, there exists no path P_i in $\mathcal{P}$ such that $u, v \in V(P_i)$.

CONJECTURE 1. Let G be a planar graph, $\mathcal{P}$ a 2-acyclic path decomposition of G and G_0 a plane graph of G. Then $\mathcal{P}$ is transversal, polygonal and colinear in G_0 if and only if there exists a geometric plane graph $\overline{G_0}$ of G induced by $\mathcal{P}$ such that G_0 is ambient isotopic to $\overline{G_0}$.

References

[1] Fáry I., On straight line representations of planar graphs, *Acta Sci. Math. (Szeged)* **11** (1948), 229-233.