Efficient Enumeration of All Ladder Lotteries

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A *ladder lottery*, known as "Amidakuji" in Japan, is a common way to choose one winner or to make an assignment randomly in Japan. Formally, a ladder lottery L of a permutation $\pi = (x_1, x_2, \ldots, x_n)$ is a network with n vertical lines (*lines* for short) and many horizontal lines (*bars* for short) connecting two consecutive vertical lines. The top ends of lines correspond to π. See Fig. 1. Each number x_i in π goes to down along the corresponding line but at each bar x_i jump to the other end of the bar. Finally, L outputs the sorted sequence $(1, 2, \ldots, n)$ at the bottom ends of lines. See Fig. 1 for an example.

A ladder lottery of a permutation $\pi = (x_1, x_2, \ldots, x_n)$ is called *optimal* if it contains no redundant bars. Let L be an optimal ladder lottery of π and m be the number of bars in L. Then we can observe that m is equal to the number of "reverse pairs" in π. For example, a ladder lottery in Fig. 1 is optimal, since a permutation (5,1,4,3,2) has seven reverse pairs: (5,1), (5,4), (5,3), (5,2), (4,3), (4,2), (3,2).

The ladder lotteries are strongly related to primitive sorting networks, which are deeply investigated by Knuth [2]. While a comparator in a primitive sorting network replaces x_i and x_{i+1} by $\min(x_i, x_{i+1})$ and $\max(x_i, x_{i+1})$, a bar in a ladder lottery always exchanges them.

In 1936 Wagner [6] showed that for any two maximal planar graphs G_1 and G_2 having the same number of vertices, one can transform G_1 and G_2 by a sequence of diagonal flip operations.

In this paper we show a similar result on ladder lotteries. Let S_π be the set of all optimal ladder lotteries of a given permutation π. A *local swap operation* is a local modification of a ladder lottery as shown in Fig. 2.

Theorem 1. *For any two optimal ladder lotteries L_1 and L_2 in S_π one can transform L_1 into L_2 by a sequence of local swap operations.*

We also give an algorithm to enumerate all optimal ladder lotteries of a given

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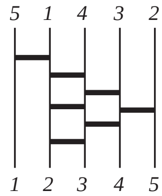

Figure 1 An optimal ladder lottery of a permutation (5,1,4,3,2).

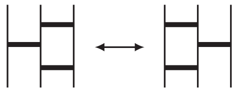

Figure 2 A local swap operation.

permutation. Our algorithm generates each ladder lottery of a permutation in $O(1)$ time in worst case. To the best of our knowledge this is the first such algorithm.

We note that the ladder lotteries for a given permutation can be regarded as a commutativity classes of the permutation for a Coxeter group of type A_{n-1} (see [5] for further details). Hence our algorithm has an application in algebraic combinatorics.

The idea of our algorithm is as follows. We first define a tree structure T_π, called *the family tree*, among ladder lotteries in S_π so that each vertex in T_π corresponds to each ladder lottery and each edge corresponds to a relation between two ladder lotteries which can be transformed from one to the other by one local swap operation. Then we design an efficient algorithm to generate all child vertices of a given vertex in T_π. Applying the algorithm recursively from the root of T_π, we can enumerate all vertices in T_π, and also corresponding ladder lotteries in S_π. Based on such tree structure but with some other ideas a lot of efficient enumeration algorithms are designed [1, 3, 4, 7].

THEOREM 2. *One can enumerate each ladder lottery in S_π in $O(1)$ time.*

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