A problem on arrangements of coins lying on the equilateral triangle latice

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This is a joint work with Kiyoshi Ando and Tomoki Nakamigawa. In this talk we will discuss a problem mentioned in a book [1]. Consider a equilateral triangle △ABC each of whose segment has length n and whose vertices are A, B and C. Mark each point on each peripheral segment S which has an integer distance from the endpoints of S, and add all straight segments passing through these points to be parallel to peripheral segments. Denote by T_n the figure given by this way and call it a *equilateraltriangle latice*. Let $V(T_n)$ be the set of the vertices of T_n. Then T_n has just $\frac{1}{2}(n+1)(n+2)$ vertices.

There are many triangles whose vertices are in $V(T_n)$. In [2], Nakamoto and Watanabe showed that the number of all triangles in T_n is given by $\lfloor \frac{1}{8}n(n+2)(2n+1)\rfloor$. A subset $H \subseteq V(T_n)$ is a *destroyer* if every three points of $V(T_n) - H$ induce no triangle.

We determined all the destroyers in $V(T_n)$ $(i \in \{1, 2, 3, 4, 5, \})$ as given by the following theorem. Let $\mathcal{D}_n$ be the set of minimum destroyers and let $\xi(n)$ be the size of a minimum destroyer. Let B_n be the arrangement a destroyer whose vertices are of $V(T_n) - \{1, 2, \cdots, 0'\} \cup \{0'', 1'', 2'', \cdots, (n-2)'', (n-1)''\}$ (see Figure 2).

THEOREM 1. *$\mathcal{D}_n = \{B_n\}$ for $i \in \{1, 2, 3, 4\}$, and $\mathcal{D}_5 = \{B_5, N_1, N_2\}$, where $B_5, N_1,$ and N_2 are the configurations as shown in* Figure 2. *Moreover, $\xi(i) = \frac{1}{2}(i^2 - i + 2)$ for $i \in \{1, 2, 3, 4, 5, \}$.*

In $T_n =$ ABC, let denote the set of the vertices of the line AB, BC, CA by S_1, S_2, S_3, respctively. The next two lemmas are necessary to discuss about T_i.

LEMMA 2. *If $n \geq 2$ and $\mathcal{D}_{n-1} = \{B_{n-1}\}$, then any $H \in \mathcal{D}_n$ satisfies*

$$|S_i \cap H| \leq n - 1 \quad \text{for } i \in \{1, 2, 3\}.$$

Moreover, if there exists $i \in \{1, 2, 3\}$ such that $|S_i \cap H| = n - 1$, then $H - S_i \equiv B_{n-1}$.

LEMMA 3. *Suppose that $2 \leq n \leq 5$. Then for any $H \in \mathcal{D}_n$ with $H \not\equiv B_n$,*

$$|S_i \cap H| \leq n - 2 \quad \text{for } i \in \{1, 2, 3\}.$$

Let $\lambda(n)$ be the maximum size of subsets K of $V(T_n)$ which any three vertices

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of K does not induce an equilateral triangle. Then, $\xi(n) + \lambda(n) = \frac{1}{2}(n+1)(n+2)$. We obtained the following:

THEOREM 4. $\lambda(n) \geq \frac{1}{6}n^{1.294}$.

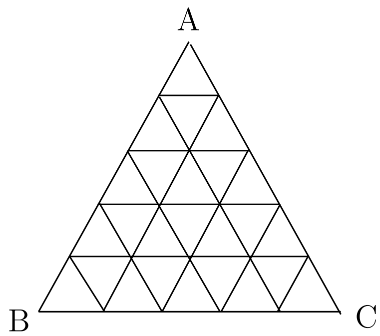

Figure 1. T_5

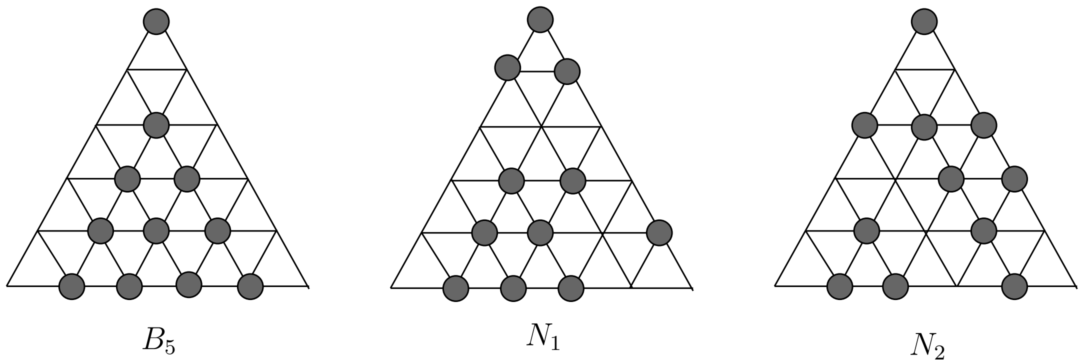

Figure 2. B_5, N_1, N_2

References

[1] Dmitri Fomin, Sergey Genkin and Ilia Itenberg, Mathematical Circles (Russian Experience), American Mathematical Society, 1992.

[2] A. Nakamoto and M. Watanabe, How many tetrahedora?, The mathematical Gazette, **86**(2002),491–498.