Balanced (C_4, C_{16})-$2t$-foil systems

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Let K_n denote *the complete graph* of n vertices. Let C_4 and C_{16} be *the* 4-*cycle* and *the* 16-*cycle*, respectively. *The* (C_4, C_{16})-$2t$-*foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_{16}'s with a common vertex. In particular, the (C_4, C_{16})-2-foil is called *the* (C_4, C_{16})-*bowtie*. When K_n is decomposed into edge-disjoint sum of (C_4, C_{16})-$2t$-foils, we say that K_n *has a* (C_4, C_{16})-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_{16})-$2t$-foils, we say that K_n *has a balanced* (C_4, C_{16})-$2t$-*foil decomposition*. This decomposition is known as *a balanced* (C_4, C_{16})-$2t$-*foil system*.

Theorem 1. *K_n has a balanced (C_4, C_{16})-$2t$-foil decomposition if and only if $n \equiv 1 \ (\ mod\ 40t)$.*

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_{16})-$2t$-foil decomposition. Let b be the number of (C_4, C_{16})-$2t$-foils and r be the replication number. Then $b = n(n-1)/40t$ and $r = (18t+1)(n-1)/40t$. Among r (C_4, C_{16})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_{16})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/40t$ and $r_2 = 18(n-1)/40$. Therefore, $n \equiv 1 \pmod{40t}$ is necessary.
(Sufficiency) Put $n = 40st + 1$, $T = st$. Then $n = 40T + 1$. Construct a (C_4, C_{16})-$2T$-foil as follows:
$\{(40T+1, 2T+1, 35T+2, 3T+1), (40T+1, 1, 4T+2, 15T+2, 28T+3, 10T+2, 24T+3, 8T+2, 17T+3, 9T+2, 26T+3, 11T+2, 30T+3, 19T+2, 6T+2, T+1)\} \cup$
$\{(40T+1, 2T+2, 35T+4, 3T+2), (40T+1, 2, 4T+4, 15T+3, 28T+5, 10T+3, 24T+5, 8T+3, 17T+5, 9T+3, 26T+5, 11T+3, 30T+5, 19T+3, 6T+4, T+2)\} \cup$
$\{(40T+1, 2T+3, 35T+6, 3T+3), (40T+1, 3, 4T+6, 15T+4, 28T+7, 10T+4, 24T+7, 8T+4, 17T+7, 9T+4, 26T+7, 11T+4, 30T+7, 19T+4, 6T+6, T+3)\} \cup ... \cup$
$\{(40T+1, 3T, 37T, 4T), (40T+1, T, 6T, 16T+1, 30T+1, 11T+1, 26T+1, 9T+1, 19T+1, 10T+1, 28T+1, 12T+1, 32T+1, 20T+1, 8T, 2T)\}$.
Decompose the (C_4, C_{16})-$2T$-foil into s (C_4, C_{16})-$2t$-foils. Then these s starters comprise a balanced (C_4, C_{16})-$2t$-foil decomposition of K_n.

Corollary 2. *K_n has a balanced (C_4, C_{16})-bowtie decomposition if and only if $n \equiv 1 \ (\ mod\ 40)$.*

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Example 1. A (C_4, C_{16})-2-foil of K_{41}.
$\{(41, 3, 37, 4), (41, 1, 6, 17, 31, 12, 27, 10, 20, 11, 29, 13, 33, 21, 8, 2)\}$.
This starter comprises a balanced (C_4, C_{16})-2-foil decomposition of K_{41}.

Example 2. A (C_4, C_{16})-4-foil of K_{81}.
$\{(81, 5, 72, 7), (81, 1, 10, 32, 59, 22, 51, 18, 37, 20, 55, 24, 63, 40, 14, 3)\} \cup$
$\{(81, 6, 74, 8), (81, 2, 12, 33, 61, 23, 53, 19, 39, 21, 57, 25, 65, 41, 16, 4)\}$.
This starter comprises a balanced (C_4, C_{16})-4-foil decomposition of K_{81}.

Example 3. A (C_4, C_{16})-6-foil of K_{121}.
$\{(121, 7, 107, 10), (121, 1, 14, 47, 87, 32, 75, 26, 54, 29, 81, 35, 93, 59, 20, 4)\} \cup$
$\{(121, 8, 109, 11), (121, 2, 16, 48, 89, 33, 77, 27, 56, 30, 83, 36, 95, 60, 22, 5)\} \cup$
$\{(121, 9, 111, 12), (121, 3, 18, 49, 91, 34, 79, 28, 58, 31, 85, 37, 97, 61, 24, 6)\}$.
This starter comprises a balanced (C_4, C_{16})-6-foil decomposition of K_{121}.

Example 4. A (C_4, C_{16})-8-foil of K_{161}.
$\{(161, 9, 142, 13), (161, 1, 18, 62, 115, 42, 99, 34, 71, 38, 107, 46, 123, 78, 26, 5)\} \cup$
$\{(161, 10, 144, 14), (161, 2, 20, 63, 117, 43, 101, 35, 73, 39, 109, 47, 125, 79, 28, 6)\} \cup$
$\{(161, 11, 146, 15), (161, 3, 22, 64, 119, 44, 103, 36, 75, 40, 111, 48, 127, 80, 30, 7)\} \cup$
$\{(161, 12, 148, 16), (161, 4, 24, 65, 121, 45, 105, 37, 77, 41, 113, 49, 129, 81, 32, 8)\}$.
This starter comprises a balanced (C_4, C_{16})-8-foil decomposition of K_{161}.

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