Every 4-connected Möbius triangulation is geometrically realizable

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Let G be a map on a surface F^2. A *geometric realization* of G is an embedding of F^2 into $\mathbb{R}^3$ such that every face of G is flat and that no two faces of G intersect at their interior. That is, a map G on a surface F^2 has a geometric realization if and only if there is a polytope which is homeomorphic to F^2 and whose 1-sckelton is isomorphic to G. Steinitz's theorem states that a map on the sphere has a geometric realization if and only if G is 3-connected.

For all surfaces, Grünbaum [6] conjectured that every triangulation on any orientable closed surface has a geometric realization. However, it was proved in 2004 that a triangulation on the orientable closed surface of genus 6 by a complete graph K_{12} has no geometric realization [3]. On the other hand, Archdeacon et al. proved in 2007 that every triangulation on the torus has a geometric realization [1].

In this talk, we consider nonorientable surfaces, in particular, the projective plane, denoted by P^2. Since no nonorientable closed surface is embeddable in $\mathbb{R}^3$, no map on it has a geometric realization. However, since the surface obtained from the projective plane by removing a disk (i.e., a Möbius band) is embeddable in $\mathbb{R}^3$, we would like to consider whether a triangulation on the Möbius band (called a *Möbius triangulation*) has a geometric realization.

However, Brehm [4] has already constructed a counterexample, that is, a Möbius triangulation with no geometric realization, which is shown in Figure 1. By this, the problem was solved negatively, but the following theorem has been proved:

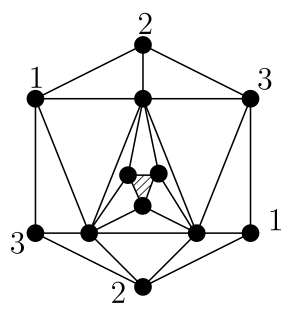

Figure 1 A Möbius triangulation, in which we identify antipodal pair of points of the hexagon, and the shaded face is removed.

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Theorem 1. (Bonnington and Nakamoto [2]) *Let G be a triangulation on the projective plane. Then G has a face f such that $G - f$ has a geometric realization.*

Analizing Brehm's example, one can see that the face f in Theorem 1 cannot be chosen in the interior of the 2-cell region of G bounded by a 3-cycle C which is disjoint from ∂f, where ∂f denotes the boundary of f. We say that such two 3-cycles C and ∂f are *nested disjoint* 3*-cycles* in G. Observe that a 5-connected triangulation has no nested disjoint 3-cycles, and hence the following theorem is natural but is slightly weak, since the connectivity seems to be decreased to 4.

Theorem 2. (Chávez, Fijavž, Márquez, Nakamoto and Suárez [5]) *Let G be a* 5*-connected triangulation on the projective plane. Then, $G - f$ has a geometric realization for any face f of G.*

So, in this talk, we shall improve Theorem 2 and characterize a face f of G whose removal gives a geometric realization of $G - f$:

Theorem 3. *Let G be a triangulation on the projective plane and let f be a face of G. Then, $G - f$ has a geometric realization if and only if G has no* 3*-cycle C forming two nested* 3*-cycles with the boundary cycle of f.*

The folowing is an immediate consequence of Theorem 3.

Corollary 4. *Every* 4*-connected Möbius triangulation is geometrically realizable.*

References

[1] D. Archdeacon, P. Bonnington and J. Ellis-Monanghan, How to exhibit toroidal maps in space, *Discrete Comp. Geom.* **38** (2007), 573–594.

[2] P.C. Bonnington and A. Nakamoto, Geometric realization of a projective triangulation with one face removed, *Discrete Comp. Geom.* **40**, 141–157 (2008).

[3] J. Bokowski and A. Guedes de Oliveira, On the generation of oriented matroids, *Discrete Comput. Geom.* **24** (2000), 197–208.

[4] U. Brehm, A non-polyhedral triangulated Möbius strip, *Proc. Amer. Math. Soc.* **89** (1983), 519–522.

[5] M.J. Chávez, G. Fijavž, A. Márquez, A. Nakamoto and E. Suárez, Geometric realization of Möbius triangulations, *SIAM J. Discrete Math*, in press.

[6] B. Grünbaum, "Convex Polytopes", *Pure and Applied Mathmatics* Vol. 16, Interscience-Wiley, New York, 1967.