Optimal 1-planar graphs which triangulate other surfaces

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A simple graph G is said to be 1-*planar* if it can be drawn on the sphere S^2(or the plane) so that each of its edges crosses at most one other edge at a point. The drawing is regarded as a continuous map $f : G \to S^2$ which may not be injective. To simplify our notation, we often consider that a given 1-planar graph G is already mapped on the sphere, and denote its image by G itself. An edge is said to be *crossing* if it crosses over another edge in a 1-planar graph G, and to be *non-crossing* otherwise. The class of 1-planar graphs has been studied from several viewpoints in many papers, for example [1, 2, 8].

For a 1-planar graph G, we have an inequality $|E(G)| \leq 4|V(G)| - 8$ where $V(G)$ and $E(G)$ stand for its vertex set and edge set, respectively. A 1-planar graph G is said to be *optimal* if it satisfies the equality $|E(G)| = 4|V(G)| - 8$. (Recently, Suzuki has discussed re-embeddability of optimal 1-planar graphs in [11].) It is also known that every optimal 1-planar graph G is obtained by adding a pair of crossing edges to each face of a 3-connected quadrangulation on the sphere. By the above fact, we have $\deg(v) \equiv 0 \pmod 2$ for any vertex v of G.

A *triangulation* of a closed surface is a simple graph embedded on the surface with no crossing edges so that each face is triangular, while quadrangulation of a closed surface is one that each face is bounded by a 4-cycle. (A k-*cycle* means a cycle of length k.) These particular embeddings have been major themes in "topological graph theory" and Negami et al. [4, 7, 9, 10] had discussed the existence of graphs having both these properties; i.e., a graph can be embedded into different closed surfaces as a triangulation and as a quadrangulation.

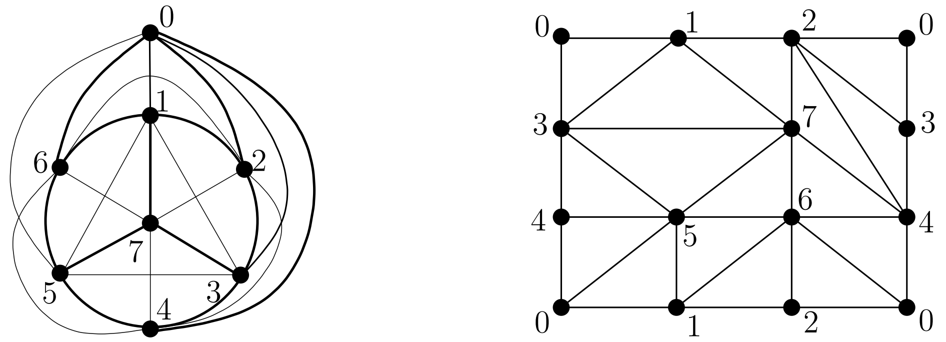

Figure 1 XW_6 triangulates the torus.

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In our research, we discuss the existence of optimal 1-planar graphs which can be embedded on other closed surfaces as triangulations. (Note that such triangulations are *even triangulations*; i.e., each vertex has even degree. Recently, Nakamoto et al. has discussed these even triangulations by focusing on a local deformation called an *N-flip*. See [3, 5, 6].) For example, see Figure 1. The left-hand side is an optimal 1-planar graph with 8 vertices, while the right-hand side is a triangulation on the torus. (To obtain the torus, identify two horizontal sides and two vertical sides of the rectangle, respectively, in the figure.) By the observation of the adjacency of those two embeddings, we can easily confirm that their abstract graphs are isomorphic to each other. If there exists such an optimal 1-planar graph G that triangulates a closed surface F^2, the number of its vertices and edges is restricted by Euler's formula as follows (where $\chi(F^2)$ is the Euler characteristic of F^2):

$$|V(G)| = 8 - 3\chi(F^2), \quad |E(G)| = 24 - 12\chi(F^2).$$

In this talk, we shall construct examples by using the method called a *slit N-flip sum* and prove the following theorem:

THEOREM 1. *Given an oriental closed surface F^2, there is an optimal 1-planar graph which can be embedded on F^2 as a triangulation.*

Furthermore, we shall discuss those graphs for nonorientable closed surfaces. However, we prove that no optimal 1-planar graph can triangulate a nonorientable surface with genus at most 3.

References

[1] R. Bodendiek, H. Schumacher, K. Wagner, Über 1-optimale Graphen, *Math. Nachr.* **117** (1984) 323–339.

[2] I. Fabrici, T. Madaras, The structure of 1-planar graphs, *Discrete Math.* **307** (2007), 854–865.

[3] K. Kawarabayashi, A. Nakamoto, Y. Suzuki, N-Flips in even triangulations on surfaces, to appear in *J. Combin. Theory, Ser. B.*

[4] A. Nakamoto, S. Negami, K. Ota, Širáň, Planar triangulations which quadrangulate other surfaces, European J. Combin. **25** (2004) 817–833.

[5] A. Nakamoto, Y. Suzuki and T. Sakuma, N-Flips in even triangulations on the sphere, *J. Graph Theory* **51** (2006), 260–268.

[6] A. Nakamoto and Y. Suzuki, N-Flips in even triangulations on the projective plane, to appear in *Discrete Math.*

[7] S. Negami, Y. Suzuki, 5-connected planar triangulations quadrangulating other surfaces. *Yokohama Math. J.*, **47**, special issue (2000), 187–193.

[8] H. Schumacher, Zur struktur 1-planarer graphen, *Math. Nachr.* **125** (1986) 291–300.

[9] Y. Suzuki, Triangulations on closed surfaces which quadrangulate other surfaces, *J. Combin. Theory, Ser. B.*, **97** (2007), 237–244.

[10] Y. Suzuki, Triangulations on closed surfaces which quadrangulate other surfaces II, *Discrete Math.*, **303** (2005), 235–243.

[11] Y. Suzuki, Re-embeddings of optimal 1-planar graphs, submitted.