Highly Symmetric Maps on Surfaces

Jozef Širáň

Slovak University of Technology, Slovakia, and Open University, UK

Abstract

We will survey the state-of-the art in the study of highly symmetric maps on compact surfaces, with emphasis on classification of regular maps on a given surface.

1 Introduction

A *map* is a cellular decomposition of a closed surface; the 1-skeleton of this cell complex is the *underlying graph* and the 2-cells of the map are its *faces*. An automorphism of the underlying graph which preserves the cell structure of the map is a *map automorphism*. The collection of all map automorphisms form the *automorphism group* of a map. The intuitive concept of 'level of symmetry' of a map can be made precise in terms of action of the automorphism group of a map on elements of the map. Thus, a map is *vertex-transitive*, *edge-transitive*, and *face-transitive* if its automorphism group acts transitively on vertices, edges, and faces, respectively.

In some sense, levels of symmetry of a map have two extremes. At the lower end one can place *Cayley maps*, in which the map automorphism group has a subgroup acting regularly on vertices. To introduce the upper end, one can see that the automorphism group of a map always acts freely on *flags*, the topological triangles whose corners are a vertex, the midpoint of an incident edge, and the centre of a face incident to the edge. The maps with the highest level of symmetry are therefore those in which the map automorphism group acts transitively, and hence regularly, on flags. Such maps are called *regular*. If the surface is is orientable and the orientation-preserving subgroup of the automorphism group of a map acts regularly on arcs, the map is said to be *orientably regular*. In an alternative terminology, regular and orientably regular maps are also known as *reflexible* and *rotary*. A map that is orientably regular but not regular (that is, rotary but not reflexible) is *chiral*.

Classification of maps with high level of symmetry (and of regular maps in particular) is an important problem due to deep connections of the theory of maps with group theory, hyperbolic geometry, Riemann surfaces, and Galois theory [10]. In the case of regular maps the three main directions of approach are classification by embedded graphs, automorphism groups, and supporting surfaces. In this extended abstract for the *20th Topological Graph Theory* workshop (Yokohama, November 2008) we will briefly comment on the development in the theory of maps with a prescribed level of transitivity and focus on the classification of regular maps on a given surface.

2 Current state and recent progress

The question of whether a given graph can be embedded on a surface in such a way that the resulting map has a prescribed level of symmetry (e.g. vertex-, edge-, or arc-transitivity) can be universally answered in terms of existence of a 'suitable' subgroup of the automorphism group of the graph, where 'suitable' means a certain type of action on the elements of the graph. Details can be found in [9, 13] and we will give more details in the presentation. The special case of Cayley maps appears to have been developed considerably [11, 5] and we will report on recent progress.

As regards regular maps, we will just briefly review the latest development in classification by underlying graphs and by automorphism groups. Focusing on classification of regular and orientably regular maps by supporting surfaces, a systematic approach to this problem was initiated by Brahana in the early 20th century by investigation of toroidal maps. Thanks to contributions of a number of authors, until the end of 1980's this was gradually extended to a classification of all chiral and regular maps on orientable surfaces of genus up to 7, and regular maps on nonorientable surfaces of genus at most 8; cf. [12] for details. A further extension was achieved computationally, covering orientable and nonorientable surfaces of genus up to 15 and 30, respectively [4]. Nevertheless, by the end of 20th century, a complete classification of regular maps was known only for finitely many (homeomorphism classes of) compact surfaces.

In the past few years there has been substantial progress progress in classification of regular maps in several directions. To indicate the nature of the results let us consider the general situation. Given any regular map of valency k and face length m on a surface of Euler characteristic χ and with automorphism group of order r, Euler's formula gives the equation

$$(km - 2k - 2m)r = -4km\chi \tag{1}$$

which is the point of departure of classification. Divisibility conditions implied by this equation tend to have powerful consequences.

The case $\chi = -p$ where p is a prime was completely solved in [2], yielding the first-ever classification of regular maps for an *infinite* set of genera (the nonorientable genera $p+2$ for prime p). As an interesting consequence of this result, there are no regular maps at all on a surface of Euler characteristic $-p$ for all primes p such that $p \equiv 1 \bmod 12$ and $p > 13$. This can be phrased in terms of presence of an infinite number of 'gaps' in the spectrum of genera of nonorientable surfaces carrying regular maps. This is quite striking since there are no such gaps in the orientable case.

From the structural point of view it is notable that the automorphism groups of regular maps of negative prime Euler characteristic are *almost Sylow-cyclic*, that is, their odd-order Sylow subgroups are cyclic and the even-order ones are dihedral. A classification of such regular maps, given in [7], also leads to a proof of non-existence of regular maps on surfaces with $\chi = -p^2$ for any prime $p > 7$.

In [3] the computational search was extended to classification of all nonorientable regular maps of genus up to 200 and all orientably regular and chiral maps of genus up to 100. The revealed new patterns in the genus spectrum of various kinds of regular maps have stimulated unprecedented progress in the structural theory of regular maps [8].

By the fundamental equation (1) the two extreme cases occur when χ divides r and when χ and r are relatively prime. For orientable surfaces, with $\chi = 2(1-g)$ where g is the genus, the two cases reduce to $g-1$ dividing r_o and $g-1$ being relatively prime to r_o where r_o is the order of the orientation-preserving part of the automorphism group of the map. In [8] we give a classification of all orientably regular maps of genus $g > 1$ such that $g-1$ is a *prime* dividing r_o, and make a major advance by producing a classification of *all* orientably regular maps of genus $g \geq 0$ for which $g-1$ is relatively prime to r_o. Corollaries include, for example, identification of primes p for which every orientably regular map of genus $p+1$ is reflexible.

3 Concluding remarks

Results of [3], [2] and [8] give a classification of all regular maps of Euler characteristic $-p$ and $-2p$ for all primes p. In the case $\chi = -2p$ we also have a classification of chiral maps in the orientable case, part of which can be found in [1].

An obvious next step is to consider regular maps on surfaces with $\chi = -tp$ for small positive integers t. Using new techniques we have classified in [6] all regular maps on surfaces of Euler characteristic $-3p$ for prime p. It is realistic to expect futher progress in this direction, in particular for prime values of t. It would also be interesting to try to extend the classification of regular maps on surfaces of Euler characteristic equal to the negative of an arbitrary power of a prime.

Acknowledgment. This research was supported by the Slovak APVV research grants No. 0040-06 and 0104-07, as well as partial support from the VEGA Research Grant No. 1/0489/08.

References

[1] Belolipetsky, M., and G.A. Jones, *Automorphism groups of Riemann surfaces of genus* $p+1$*, where* p *is prime*, Glasgow Math. J. **47** (2005), 379–393.

[2] Breda d'Azevedo, A., R. Nedela, and J. Širáň, *Classification of regular maps with a negative prime Euler characteristic*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4175–4190.

[3] Conder, M., *Regular maps and hypermaps of Euler characteristic* -1 *to* -200, Preprint (2006), submitted for publication.

[4] Conder, M. and P. Dobcsányi, *Determination of all regular maps of small genus*, J. Combinat. Theory Ser. B **81** (2001), 224–242.

[5] M.D.E. Conder, R. Jajcay, and T.W. Tucker, Regular Cayley maps for abelian groups, *J. Algebraic Combinatorics* 25 (2007), 259-283.

[6] Conder, M., R. Nedela, and J. Širáň, *Regular maps of nonorientable genus* $3p+2$*, where* p *is a prime*, Manucript in preparation.

[7] Conder, M., P. Potočnik, and J. Širáň, *Regular maps with almost Sylow-cyclic automorphism groups, and classification of regular maps with Euler characteristic* $-p^2$, Preprint (2007), submitted for publication.

[8] Conder, M., J. Širáň, and T. W. Tucker, *The genera, reflexibility and simplicity of regular maps*, Preprint (2007), submitted for publication.

[9] A. Gardiner, R. Nedela, J. Širáň and M. Škoviera, Characterization of graphs which underlie regular maps on closed surfaces, J. London Math. Soc. (2) 59 (1999) No. 1, 100–108.

[10] Jones, G. A., and D. Singerman, *Belyĭ functions, hypermaps, and Galois groups*, Bull. London Math. Soc. **28** (1996), 561–590.

[11] B. Richter, j. Širáň, R. Jajcay, M. Watkins, T. Tucker, *Cayley maps*, J. Combinat. Theory Ser. B. **92** (2005) No. 2, 189-245.

[12] Širáň, J., *Regular maps on a given surface: A survey*, in: "Topics in Discrete Mathematics", Springer Series **26** (2006), 591–609.

[13] J. Širáň and T. Tucker, *Characterization of graphs which admit vertex-transitive embeddings*, J. Graph Theory **55** (2007) 3, 233-248.