Covalences and quotients of Cayley tessellations

Jana Šiagiová*

A *Cayley tessellation* $\mathcal{T}$ is an infinite tessellation [1] of the euclidean or hyperbolic plane (or, equivalently, an infinite graph embedded on such a plane with all faces finite) for which there exists a group of direct euclidean or hyperbolic isometries G that preserves $\mathcal{T}$ and acts regularly on the vertex set of $\mathcal{T}$. The group G is also called a *Cayley group* for $\mathcal{T}$.

Consider the quotient $\mathcal{M} = \mathcal{T}/G$, which is a map on a compact, connected, orientable surface. By regularity of G on vertices of $\mathcal{T}$, the underlying graph of $\mathcal{M}$ has just one vertex, the valence of which is equal to the valence of $\mathcal{T}$. We call $\mathcal{M}$ the *single-vertex quotient* of $\mathcal{T}$. The projection $\mathcal{T} \to \mathcal{M}$ is a regular covering, unbranched at the single vertex of $\mathcal{M}$ but with possible branch points in centers of faces of $\mathcal{M}$ and in midpoints of edges.

The *covalence sequence* τ of a Cayley tessellation $\mathcal{T}$ is the cyclic sequence of covalences (that is, lengths of facial walks) appearing around a (and hence any) vertex in the clockwise direction. We also say that the Cayley tessellation $\mathcal{T}$ *realizes* the sequence τ. A characterization of covalence sequences of Cayley tessellations was given in [3]. We will be primarily interested in the study of covalence sequences of Cayley tessellations in relation with the genera of their quotient maps.

If a given covalence sequence τ is realizable by a Cayley tessellation, there may exist a number of non-isomorphic Cayley tessellations $\mathcal{T}$ the covalence sequence of which is τ. Moreover, each such $\mathcal{T}$ may admit several non-isomorphic Cayley groups. This way a given covalence sequence may be related to a number of groups and to a number of quotient surfaces (and hence to a number of genera). We therefore define the *quotient genus* of the covalence sequence τ to be the smallest genus of the supporting surface of the quotient map $\mathcal{T}/G$ where G ranges over all possible Cayley groups of all Cayley tessellations $\mathcal{T}$ realizing the sequence.

The quotient genus turns out to be bounded above by the length of the covalence sequence. More specifically, if τ is a covalence sequence of a Cayley tessellation such that the length of τ (and hence the degree of the tessellation) is ℓ, then the quotient genus g of τ satisfies $g \le \lfloor(\ell-4)/4\rfloor$. This upper bound is shown to be tight for infinitely many ℓ's. We will focus on the cases when the quotient embedding on a surface of genus $\lfloor(\ell-4)/4\rfloor$ has exactly two or three faces. The reason is that quotient embeddings on such surfaces with at least four faces would necessarily result in longer covalence sequences. This can be equivalently stated as follows.

*Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia. E-mail: siagiova@math.sk

Theorem 1. *A covalence sequence of length $4g+4$ with at least four distinct entries has quotient genus smaller than g.*

Among the results that we would like to present are also the following two theorems.

Theorem 2. *Let p and q be twin primes such that $p = 2g+1$ and $q = 2g+3$ for some $g \geq 1$. Then there exists a covalence sequence τ of a Cayley map of valence $4g+4$ such that τ contains exactly two distinct odd covalences (which are certain multiples of p and q) and the quotient genus of τ is g.*

Theorem 3. *Let $p = 2g+1$ be a prime. Then there exists a Cayley tessellation with covalence sequence τ of length $4g+4$ with one even and two distinct odd covalences (certain multiples of p), such that the quotient genus of τ is g.*

At the other end of the spectrum, of special interest are the Cayley tessellations of quotient genus equal to 0. Examples of such results are:

Theorem 4. *Let $\tau = (a_1^{k_1} b^{\ell_1} a_2^{k_2} b^{\ell_2} \dots a_t^{k_t} b^{\ell_t})$ be a covalence sequence of a Cayley tessellation consisting of entries $a_1, a_2, \dots, a_t, b$ such that for each i, $1 \leq i \leq t$, either each a_i is even or $k_i = 1$. If $\sum_{i=1}^{t} \ell_i$ divides b, then the quotient genus of τ is equal to* 0.

Corollary 5. *Let τ be a covalence sequence of a Cayley tessellation such that τ has just two distinct entries a and b, and either a is even or the subsequence aa does not appear in τ. If the total number of entries b appearing in τ divides b, then the quotient genus of τ is* 0.

We conclude with a statement relating Cayley tessellations with Cayley maps; for a general theory of Cayley maps we refer to [2].

Theorem 6. *A sequence τ is a covalence sequence of a Cayley tessellation if and only if τ is a covalence sequence of a Cayley map on a compact surface of genus at least* 1.

References

[1] B. Grünbaum and G. C. Shephard, *"Tilings and Patterns"*, W. H. Freeman and Co. New York, 1987.

[2] R. B. Richter, J. Širáň, R. Jajcay, T. W. Tucker, and M. E. Watkins, Cayley maps, *J. Combin. Theory Ser. B* 95 (2005), no. 2, 189-245.

[3] Šiagiová, J. and Watkins, M. E.: Covalence sequences of planar vertex-homogeneous maps, *Discrete Mathematics* 307 (2007), 599-614.