Spatial graph diagrams whose edges have no self-crossings

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We give a natural extension to spatial graphs of the result given by J. H. Lee and G. T. Jin for link diagrams. As an application, we prove that any loopless spatial graph has a diagram where each edge has no self-crossings.

In this talk, we work in the piecewise linear category. By a *graph* we mean a finite graph. We call an embedding f of a graph Γ in the 3-dimensional space $\mathbb{R}^3$ a *spatial embedding* of Γ, and the image $f(\Gamma)$ is called a *spatial graph*. In particular, $f(\Gamma)$ is called a *knot* (resp. *link*) if Γ is homeomorphic to a circle (resp. a disjoint union of circles). A *subgraph* of Γ is a graph that has a subset of vertices and a subset of edges with respect to Γ. A *spatial subgraph* of $f(\Gamma)$ is a spatial graph which is the image of a spatial embedding f restricted on a subgraph of Γ. In particular if $f(\Gamma)$ is a link, a spatial subgraph of $f(\Gamma)$ is called a *sublink*. We denote the spatial subgraph of a spatial graph G consisting of all vertices of G by $V(G)$. Two spatial graphs G and G' are said to be *equivalent*, denoted $G \sim G'$, if there is an orientation preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(G) = G'$.

A *diagram* D of a spatial graph G is the planar image of a spatial graph which is equivalent to G, under the projection $\mathbb{R}^3 \to \mathbb{R}^2 \equiv \mathbb{R}^2 \times 0$ having only finitely many transverse double points away from vertices with crossing information specifyied. Two diagrams D and D' are said to be *equivalent*, denoted by $D \sim D'$, if there is an orientation preserving homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(D) = D'$. Two spatial graphs G_1 and G_2 are said to be *edge disjoint* if $G_1 \cap G_2 \subset V(G_1) \cap V(G_2)$. Let D be a diagram of a spatial graph G. The diagram of a spatial subgraph H of G obtained from D by removing the edges and vertices not belonging to H will be called the *restriction* of D to H and denoted by $D(H)$. A *loop* is an edge which starts and ends on the same vertex.

For link diagrams, Lee and Jin [1] gave the following theorem.

Theorem 1. *Let $D_1, D_2, \ldots, D_n$ are diagrams and L a link which is partitioned into sublinks $L_1, L_2, \ldots, L_n$ admitting diagrams $D_1, D_2, \ldots, D_n$, respectively. Then there is a diagram D of L whose restictions to $L_1, L_2, \ldots, L_n$ are equivalent to $D_1, D_2, \ldots, D_n$, respectively.*

We generalize Theorem 1 to spatial graph diagrams as follows.

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Theorem 2. *Let G be a spatial graph which is a union of edge disjoint spatial graphs $H_1, H_2, \ldots, H_n$ admitting diagrams $D_1, D_2, \ldots, D_n$, respectively. Then there is a diagram D of G such that $D(H_i) \sim D_i$ $(i = 1, 2, \ldots n)$.*

If G is a link, Theorem 2 coincides with Theorem 1. Here we give an example of Theorem 2.

Example 1. Let G be a spatial graph as in Figure 1. The spatial graph G is a union of two edge disjoint spatial graphs H_1 and H_2. From the figure, it is clear that H_1 and H_2 admit the diagram D_1 and D_2, respectively. Since the spatial graph G' in the figure is equivalent to G, the diagram D obtained from G' is a diagram of G. We can see that the restrictions of D to H_1 and H_2 are equivalent to D_1 and D_2, respectively. Hence D is a diagram of G which satisfies the condition of Theorem 1.

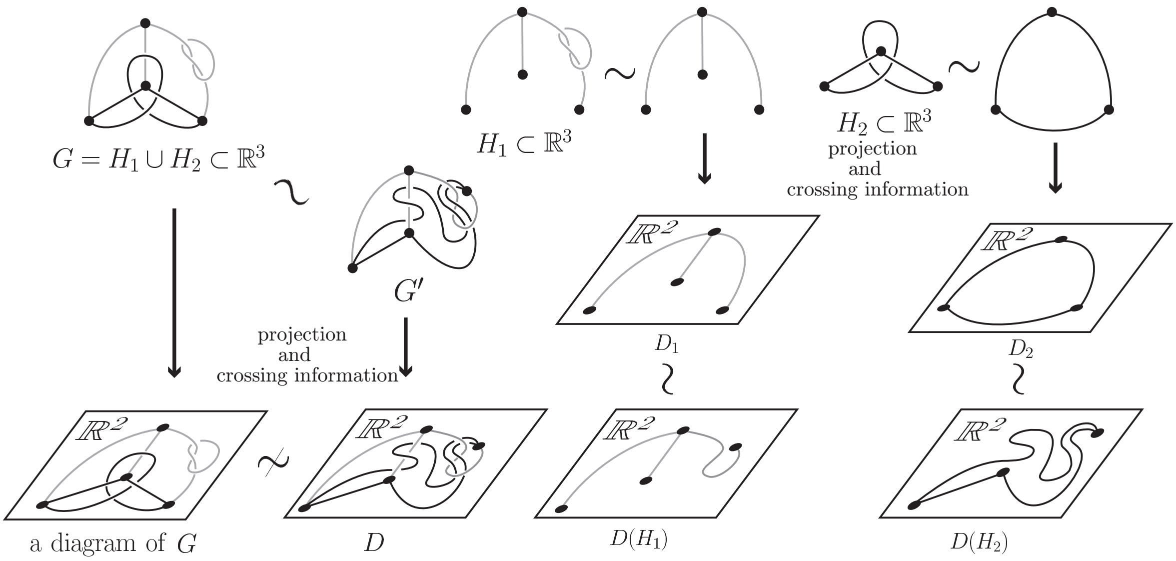

Figure 1 An example of Theorem 1

As an application of Theorem 2, we obtain the following corollary.

Corollary 3. *Let G be a loopless spatial graph. Then G has a diagram in which no edges have self-crossings.*

References

[1] J. H. Lee and G. T. Jin: Link diagrams realizing prescribed subdiagram partitions, Kobe J. Math., 18 (2001), 199–202.