Diagonal flips in outer-torus triangulations

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A *triangulation* on a surface is a map of a simple graph on the surface such that each face is bounded by a 3-cycle. A *diagonal flip* in a triangulation G is an operation replacing an edge e with another diagonal in the quadrilateral region formed by two triangular faces sharing e If the map obtained from G by a diagonal flip is not simple, then we do not apply it. Two triangulations G an G' are said to be *equivalent* if G and G' can be transformed into each other by a sequence of diagonal flips.

Wagner proved that any two triangulations on the sphere with the same number of vertices are equivalent [7]. For the torus, the projective plane and the Klein bottle, Dewdney [1], Negami and Watanabe [6] proved the same facts. Moreover, Negami [5] proved that for any surface F^2, there exists an integer $N(F^2)$ such that any two triangulations on F^2 are equivalent if they have the same and sufficiently large number of vertices.

In my talk, we would like to consider the number of diagonal flips needed to transform one triangulation into the other in the above results. For two spherical triangulations with n vertices, through the algorithm of Wagner's proof requires $O(n^2)$ diagonal flips, Mori et al. proved the following theorem, focusing on the fact that every 4-connected spherical triangulation G can be decomposed into two spanning maximal outer-plane graphs sharing a hamiltonian cycle C.

Theorem 1. (Mori, Nakamoto and Ota [3]) *Any two spherical triangulations with n vertices are equivalent by at most $6n - 30$ diagonal flips.*

Let F^2 be a surface. An *outer-F^2 triangulation* is a map on F^2 if it has exactly one face f bounded by a hamiltonian cycle and all other faces are triangular. (For example, an outer-sphere triangulation is a maximal outer-plane graph.) Mori and Nakamoto proved the following, focusing that every 4-connected triangulation G is separated into two spanning subgraphs sharing a contractible hamitonian cycle C: one is an outer-projective-plane triangulation and the other is an outer-sphere triangulation.

Theorem 2. (Mori and Nakamoto [4]) *Any two triangulations on the projective plane with n vertices are equivalent by at most $8n - 26$.*

To prove Theorem 2, they proved that any two outer-projective-plane triangu-

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lations are equivalent by at most $4n-6$ diagonal flips.

Our main theorem is the following for the torus, where the equivalence of two outer-torus triangulations has already been solved in [2].

Theorem 3. *Any two outer-torus triangulations with n vertices are equivalent by at most $8n-34$ diagonal flips, where c is a constrant.*

We conjecture the following. In order to prove it, we have only to prove that every toroidal triangulation can be transformed into a triangulation with a contractible hamiltonian cycle by at most $O(n)$ diagonal flips, or that every 4-connected toroidal graph has a contractible hamiltonian cycle. (The latter is related to a famous conjecture by Nash-Willians.)

Conjecture 1. Any two triangulations on the torus with n vertices are equivalent by at most $O(n)$ diagonal flips.

References

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