Perron-Frobenius matrices of finite graphs and edge zeta functions

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Let $X = (V, E)$ be a finite undirected loop-free graph with vertex set $V = \{v_1, \ldots, v_m\}$ and edge set $E = \{e_1, \ldots, e_n\}$. Let $\mathbf{D}(X^{\mathsf{o}})$ be a symmetric digraph uniquely constructed from X^{o} with a direction o as follows:

$$\mathbf{D}(X^{\mathsf{o}}) := (V, E''), \quad E'' := \{e''_{(i,1)}, e''_{(i,2)} : 1 \leq i \leq n\},$$

with $e''_{(i,1)} := e_i^{\mathsf{o}}$ and $e''_{(i,2)} := \overline{e_i^{\mathsf{o}}}$ for $1 \leq i \leq n$, where $\overline{e^{\mathsf{o}}}$ indicates the inverse arc of e^{o}. The **Perron-Frobenius matrix** $T = T(X^{\mathsf{o}}) = (t_{i,j})$ of X^{o} is defined by

$$t_{i_1+n(j_1-1), i_2+n(j_2-1)} := \begin{cases} 1, & \text{if } e''_{(i_1,j_1)} \text{ feeds into } e''_{(i_2,j_2)} \text{ in } \mathbf{D}(X^{\mathsf{o}}) \text{ and } i_1 \neq i_2, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \leq i_1, i_2 \leq n$ and $1 \leq j_1, j_2 \leq 2$.

The Perron-Frobenius matrix of X^{o} is important to compute the edge zeta functions ζ_X of X due to the following formula ([1], [4]):

$$\zeta_X(u_1, \ldots, u_n)^{-1} = \det(I_{2n} - UT) = \det(I_{2n} - TU), \tag{1}$$

where $U = \mathrm{diag}(u_1, \ldots, u_n, u_1, \ldots, u_n)$. Note that ζ_X is independent of the direction o of the graph X^{o}. The reciprocal $Z_X(u)^{-1}$ of the Ihara zeta function can be obtained by specializing variables $u_i = u$ for all $i = 1, \ldots, n$ in (1).

We give a direction $\sigma = \sigma(X)$ to X determined by the labelling of X as follows:

$$i < j \iff o(e^\sigma) = v_i, \quad t(e^\sigma) = v_j,$$

for all $e = \{v_i, v_j\} \in E$, where $o(e^\sigma)$ (resp. $t(e^\sigma)$) is the origin (resp. terminus) of an arc $e^\sigma \in E^\sigma$.

Definition 1. For a graph X, we define a bipartite graph $\mathbf{B} = \mathbf{B}(X) := (V', E')$ as follows. Let $V' = \{v_1, \ldots, v_{m+n}\} = V \cup E$ with $v_{m+1} := e_1, \ldots, v_{m+n} := e_n$ and

$$E' := \{f_{(i,1)}, f_{(i,2)} : 1 \leq i \leq n\},$$

where $f_{(i,1)} := \{e_i, v_{j_1}\}$ and $f_{(i,2)} := \{e_i, v_{j_2}\}$ satisfy $e_i = \{v_{j_1}, v_{j_2}\} \in E$ and $j_1 < j_2$. Here we remark that $j_1 \neq j_2$ because X is loop-free, hence the above labelling of the edges of $\mathbf{B}$ is well-defined. We index every edge of $\mathbf{B}$ by $e'_{2i+j-2} := f_{(i,j)}$.

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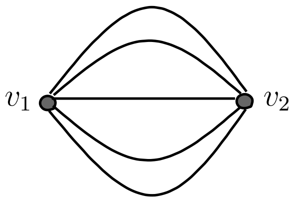

Figure 1 Γ_5

THEOREM 1. *Let X be a finite graph and $\mathbf{B} = \mathbf{B}(X)$ be the associate bipartite graph with the labelling of edges defined as in Definition 1. Let T be the Perron-Forbenius matrix of X^σ and $n = |E|$. Then the following formula holds:*

$$T = PA(\mathbf{B}_L)N\,{}^{\mathsf{T}}P - I_{2n},$$

where $\mathbf{B}_L$ is the undirected line graph of $\mathbf{B}$, $P := (p_{ij})$ is a $2n \times 2n$ permutation matrix defined by:

$$p_{ij} := \begin{cases} 1, & \text{if } i = n(j+1) - (2n-1)\lfloor (j+1)/2 \rfloor, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le i, j \le 2n$, ${}^{\mathsf{T}}P$ denotes the transposed matrix P, and N is the $2n \times 2n$ block diagonal matrix of the form:

$$N := \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_1 \end{bmatrix}, \qquad N_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

COROLLARY 2. *Let $n \in \mathbb{Z}_{>0}$, and let Γ_n be the graph with vertex set $V = \{v_1, v_2\}$ and edge set $\{e_1, \ldots, e_n\}$ such that two vertices of e_i are v_1 and v_2 for all $1 \le i \le n$. Then the reciprocal of the edge zeta function of Γ_n is*

$$\zeta_{\Gamma_n}^{-1} = \left(1 - \sum_{\ell=1}^{n} (\ell - 1) \sum_{(i_1,\ldots,i_\ell)} u_{i_1} \ldots u_{i_\ell}\right)\left(1 - \sum_{\ell=1}^{n} (-1)^{\ell-1} (\ell - 1) \sum_{(i_1,\ldots,i_\ell)} u_{i_1} \ldots u_{i_\ell}\right),$$

where the second summation runs through all ℓ-tuples $(i_1, \ldots, i_\ell) \in \{1, 2, \ldots, n\}^\ell$ such that $i_1 < \cdots < i_\ell$.

References

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