Bartholdi zeta functions for hypergraphs

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1. Introduction

Zeta functions of graphs were studied by Ihara, Sunada, Hashimoto, Bass and Bartholdi.

Storm defined the Ihara-Selberg zeta function of a hypergraph. A hypergraph $H = (V(H), E(H))$ is a pair of a set of *hypervertices* $V(H)$ and a set of *hyperedges* $E(H)$, which the union of all hyperedges is $V(H)$. A hypervertex v is *incident* to a hyperedge e if $v \in e$. For a hypergraph H, its dual H^* is the hypergraph obtained by letting its hypervertex set be indexed by $E(H)$ and its hyperedge set by $V(H)$. A bipartite graph B_H associated with a hypergraph H is defined as follows: $V(B_H) = V(H) \cup E(H)$ and $v \in V(H)$ and $e \in E(H)$ are *adjacent* in B_H if v is incident to e. Let $V(H) = \{v_1, \ldots, v_n\}$. Then an *adjacency matrix* $\mathbf{A}(H)$ of H is defined as a mtarix whose rows and columns are parameterized by $V(H)$, and (i, j)-entry is the number of directed paths in B_H from v_i to v_j of length 2 with no backtracking.

Let H be a hypergraph. A *path P of length n* in H is a sequence $P = (v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1})$ of $n + 1$ hypervertices and n hyperedges such that $v_i \in V(H)$, $e_j \in E(H)$, $v_1 \in e_1$, $v_{n+1} \in e_n$ and $v_i \in e_i, e_{i-1}$ for $i = 2, \ldots, n - 1$. Set $\mid P \mid = n$, $o(P) = v_1$ and $t(P) = v_{n+1}$. Also, P is called an *$(o(P), t(P))$-path*. We say that a path P has a *hyperedge backtracking* if there is a subsequence of P of the form (e, v, e), where $e \in E(H)$, $v \in V(H)$. A (v, w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$.

We introduce an equivalence relation between cycles. Such two cycles $C_1 = (v_1, e_1, v_2, \cdots, e_m, v_1)$ and $C_2 = (w_1, f_1, w_2, \cdots, f_m, w_1)$ are called *equivalent* if $w_j = v_{j+k}$ and $f_j = e_{j+k}$ for all j. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no hyperedge backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. The *Ihara-Selberg zeta function* of H is defined by $\zeta_H(t) = \prod_{[C]}(1 - t^{|C|})^{-1}$, where $[C]$ runs over all equivalence classes of prime, reduced cycles of H, and t is a complex variable with $\mid t \mid$ sufficiently small.

Let H be a hypergraph with $E(H) = \{e_1, \ldots, e_m\}$, and let $\{c_1, \ldots, c_m\}$ be a

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set of m *colors*, where $c(e_i) = c_i$. Then an *edge-colored graph* GH_c is defined as a graph with vertex set $V(H)$ and edge set $\{vw \mid v, w \in V(H); v, w \in e \in E(H)\}$, where an edge vw is colored c_i if $v, w \in e_i$. Let GH^o_c be the symmetric digraph corresponding to the edge-clored graph GH_c. Then the *oriented line graph* $H^o_L = (V_L, E^o_L)$ associated with GH^o_c by $V_L = D(GH^o_c)$, and $E^o_L = \{(e_i, e_j) \in D(GH^o_c) \times D(GH^o_c) \mid c(e_i) \neq c(e_j), t(e_i) = o(e_j)\}$, where $c(e_i)$ is the color assigned to the oriented edge $e_i \in D(GH^o_c)$. The *Perron-Frobenius operator* $T : C(V_L) \longrightarrow C(V_L)$ is given by $(Tf)(x) = \sum_{e \in E_o(x)} f(t(e))$, where $E_o(x) = \{e \in E^o_L \mid o(e) = x\}$ is the set of all oriented edges with x as their origin vertex, and $C(V_L)$ is the set of functions from V_L to the complex number field $\mathbf{C}$.

Storm gave two nice determinant expressions of the Ihara-Selberg zeta function of a hypergraph by using the results of Kotani and Sunada, and Bass.

Theorem 1. (Storm) *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then* $\zeta_H(t)^{-1} = \det(\mathbf{I} - tT) = (1 - t)^{m-n} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + t\mathbf{Q}_{B_H})$, *where* $n =\mid V(B_H) \mid$, $m =\mid E(B_H) \mid$ *and* $\mathbf{Q}_{B_H} = \mathbf{D}_{B_H} - \mathbf{I}$.

Furthermore, Storm presented the Ihara-Selberg zeta function of a (d, r)-regular hypergraph by using the result of Hashimoto.

We define the Bartholdi zeta function of a hypergraph, and present a determinant expression of it. Furthermore, we give a decomposition formula for the Bartholdi zeta function of semiregular bipartite graph.

2. Bartholdi zeta function of a hypergraph

Let H be a hypergraph. Then a path $P = (v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1})$ has a *(broad) backtracking* or *(broad) bump* at e or v if there is a subsequence of P of the form (e, v, e) or (v, e, v), where $e \in E(H)$, $v \in V(H)$. Furthermore, the *cyclic bump count* $cbc(C)$ of a cycle $C = (v_1, e_1, v_2, e_2, \cdots, e_n, v_1)$ is $cbc(C) =\mid \{i = 1, \cdots, n \mid v_i = v_{i+1}\} \mid + \mid \{i = 1, \cdots, n \mid e_i = e_{i+1}\} \mid$, where $v_{n+1} = v_1$ and $e_{n+1} = e_1$. The *Bartholdi zeta function* of H is defined by $\zeta(H, u, t) = \prod_{[C]}(1 - u^{cbc(C)}t^{|C|})^{-1}$, where $[C]$ runs over all equivalence classes of prime cycles of H, and u, t are complex variables with $\mid u \mid$, $\mid t \mid$ sufficiently small. If $u = 0$, then the Bartholdi zeta function of H is the Ihara-Selberg zeta function of H.

Theorem 2. *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then* $\zeta(H, u, t) = \zeta(B_H, u, \sqrt{t}) = (1-(1-u)^2 t)^{-(m-n)} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + (1-u)t(\mathbf{D}_{B_H} - (1-u)\mathbf{I}))^{-1}$, *where* $n =\mid V(B_H) \mid$ *and* $m =\mid E(B_H) \mid$.

If $u = 0$, then Theorem 2 implies Theorem 1.