Panel-irreducible triangulations on the Klein bottle

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We shall deal with simple graphs. An *embedding* of a graph G into a closed surface F^2 is an injective continuous map $f\colon G \to F^2$ which induces a graph isomorphism between G and $f(G)$, regarding G as a topological space. Two embeddings $f, f'\colon G \to F^2$ are said to be *equivalent* to each other if there exists a homeomorphism $h\colon F^2 \to F^2$ with $hf = f'$.

Then the following problems arise naturally:

- How many inequivalent embeddings into a closed surface F^2 does a graph G have?
- What structure generates inequivalent embeddings of G? (Such a structure is often called the *re-embedding structure* of G.)

However, it is hard to answer these questions for general graphs. Nevertheless, if the graphs are "triangulations" on closed surfaces, the problems become relatively simple.

A simple graph embedded on a closed surface F^2 is called a *triangulation* if it is embedded on F^2 so that every face is bounded by a 3-cycle and any two faces share at most one edge. As for the above questions for the triangulations, Whitney [6] showed that any 3-connected planar graph has a unique dual, which implies that any triangulation on the sphere S^2 is uniquely embedded on S^2, up to equivalence. On the other hand, Lawrencenko [1] discussed the re-embedding structures of triangulations on the projective plane and proved the following theorem.

Theorem 1. (Lawrencenko [1]) *Every triangulation on the projective plane has precisely 1–4, 6 or 12 inequivalent embeddings.*

Furthermore, the author determined all the re-embedding structures of triangulations on the torus and proved the following theorem.

Theorem 2. (Sasao [4]) *Every triangulation on the torus has precisely 1–6, 8, 10, 12, 14, 16, 24, 48 or 120 inequivalent embeddings.*

So, as the next step, we would like to study the re-embedding structures of triangulations on the Klein bottle. To do this, we shall apply the notions called the "panel structure" of a triangulation and the "paneled triangulation", which are introduced by Negami et al. [3]. Let G be a triangulation on a closed surface F^2.

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A 3-cycle C of G is called a *panel* if $f(C)$ bounds a face of $f(G)$ for any embedding $f\colon G \to F^2$. A face bounded by such a 3-cycle is also called a *panel* of G. The set of panels of G is denoted by $\wp(G)$. The composite structure $(G, \wp(G))$ is called the *panel structure* of G.

It is clear that inequivalent embeddings of G are generated by the 3-cycles which do not belong to $\wp(G)$. Therefore, once we know the panel structure of G, we would be able to determine the re-embedding structure of G and the number of its inequivalent embeddings. The *paneled triangulation* is an artificial object to obtain the panel structure of a triangulation, which represents the equivalence class of panel structures.

Of course, there exist infinitely many triangulations on a closed surface. However, it is shown in [3] that the panel structure of every triangulation is equivalent to one of the panel structures of "panel-irreducible saturated paneled triangulations" ("panel-irreducible triangulations", for short), which are finite in number. Therefore, if we obtain the triangulations on the Klein bottle K^2 which underlie them, we can determine all the equivalence classes of panel structures of triangulations on K^2 and all the number of inequivalent embeddings which every triangulation on K^2 can attain. That is the aim of this talk, and the following is our main theorem.

Theorem 3. *There exist precisely 39 triangulations which underlie the panel-irreducible triangulations on the Klein bottle. Among them, 29 ones are the irreducible triangulations on the Klein bottle determined by Lawrencenko and Negami [2] and Sulanke [5], and the other 10 are not irreducible ones determined by the author.*

We can obtain the similar result to Theorems 1 and 2 for the Klein bottle by the above 39 triangulations. If possible, the theorem for the Klein bottle would be reported.

References

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