Distinguishing chromatic numbers of planar graphs

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Let G be a graph. A assignment $c : V(G) \to \{1, 2, \ldots, d\}$ of d numbers (or "colors") to the vertice is said to be *d-distinguishing* if the only automorphism σ of G with $c(\sigma(v)) = c(v)$ for all vertices $v \in V(G)$ is the identity map over G. A graph G is said to be *d-distinguishable* if G admits a d-distinguishing assignment not assumed to be a proper coloring. The *distinguishing number* of G is defined as the minimum number d such that G becomes d-distinguishable and is denoted by $D(G)$.

This notion of distinguishing number has been discussed in many papers [1, 2] and so on. Also there has been studies on the distinguishing number of graph embedded on surfaces [4, 5, 6]. In particular, Fuckuda, Negami and Tucker have established the following theorem on the distinguishing number of planar graphs, which can be said to be a starting point of a general theory, developed by Negami in [5], to analyze the distinguishing number of graphs on surfaces:

THEOREM 1. (Fukuda, Negami and Tucker [4]) *Every 3-connected planar graph is 2-distinguishable, except* K_4, $K_{2,2,2}$, W_4, W_5, $C_3 + \overline{K_2}$, $C_5 + \overline{K_2}$ *and* Q_3.

It is easy to determine the distinguishing number of the seven exceptions:

$$D(K_4) = 4, D(K_{2,2,2}) = D(W_4) = D(W_5) = D(C_3+\overline{K_2}) = D(C_5+\overline{K_2}) = D(Q_3) = 3$$

On the other hand, a graph G is said to be *d-distinguishing colorable* if G has a d-distinguishing coloring, which is a d-distinguishing assignment such that any adjacent pair of vertices get different colors. The *distinguishing chromatic number* $\chi_D(G)$ is defined as the minimum number d such that G is d-distinguishing colorable; this definition can be found in [3]. It is obvious that $D(G) \le \chi_D(G)$. For example, it is not difficult to see that $D(K_n) = \chi_D(K_n) = n$ and that $D(K_{n,n,n}) = n + 1 < \chi_D(K_{n,n,n}) = 3n$ for $n \ge 2$.

It is well-known that every planar graph is 4-colorable, as "Four Color Theorem". This fact suggests that the distinguishing number of planar graphs is not so big, that is, there is an upper bound for it. Corresponding to Theorem 1, we shall show the following theorem:

THEOREM 2. *Every 3-connected planar graph is 6-distinguishing colorable.*

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A graph is said to be *maximal planar* if it is embedded on the plane and if adding any new edge yields a nonplanar graph. A maximal planar graph is often called a triangulation on the plane or the sphere since each face is triangular. The chromatic number of a maximal planar graphs is larger than that of others, but the opposite phenomenon seems to heppen for the distinguishing chromatic number.

Theorem 3. *Every maximal planar graph is* 5*-distinguishing-colorable unless it is isomrphic to* $K_{2,2,2}$ *or* $C_6 + \overline{K}_2$.

Note that there exist a series of 2-connected planar graphs whose distinguishing chromatic numbers become arbitrarily large. For example, we have:

$$D(K_{2,n}) = n, \quad \chi_D(K_{2,n}) = 2 + n \quad (n \geq 3)$$

References

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