Path-Parity-Solvability and its Heredity

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1. Introduction

Let G be a simple graph, (p, q) a non-adjacent vertex-pair of G. Then the pair (p, q) is called an *even-pair* (*odd-pair*, resp.) if every induced path from p to q of G is even (odd, resp.) length. The pair (p, q) is called a *broken-pair* if G has two induced paths connecting them with different parities, a *disconnected pair* if there is no path connecting p and q in G. The *path-parity problem* on G is as follows:

Problem 1.1 (The Path-Parity Problem)

INSTANCE:	A simple graph G and its non-adjacent pair of vertices (p, q).
TASK:	Determine whether the pair (p, q) is an even-pair or an odd-pair or a broken pair or a disconnected pair.

D. Bienstock [1] proved that this problem is NP-hard in general. Actually he proved much stronger result: *It is coNP-complete to determine whether a given general graph has an even-pair.* On the other hand, polynomially-time algorithms for the path-parity problem are found on perfect graphs [3].

2. PPS-Hereditary

Let $\mathcal{X}$ be a graph class. Then $\mathcal{X}$ is *Path-Parity-Solvable* (*PPS*, for short,) if there exists a positive real number t such that we can make a unified algorithm adaptable to every graph $G \in \mathcal{X}$ which solves the Path-Parity-Problem for any prescribed non-adjacent vertex-pair of G in $O(|V(G)|^t)$-time. A graph class $\mathcal{X}$ is *PPS-Hereditary* if it is Path-Parity-Solvable and is also hereditary property of graphs (that is, $\forall G \in \mathcal{X}$, every induced subgraph of G is also in $\mathcal{X}$).

One of our main motivation for the study of PPS-Hereditary may be the odd-hole recognition problem. The time complexity of the odd-hole recognition problem is still unknown (however, some experts believe its polynomially time solvability (see [3, 5])), and we can safely say that it is one of the most important algorithmic issue associated with the famous Strong Perfect Graph Theorem [2], because, if such a polynomially time recognition algorithm exists, it will be a good generalization of the polynomially time solvability of the recognition of perfectness [3].

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To determine the time complexity of odd-hole recognition problem is fairly difficult matter. At present, polynomially time recognition algorithm is found only for the graphs of bounded clique number [4]. Hence it is natural to investigate and try various approaches to attack this hard problem. We can obtain one of such the approaches by combining the following two theorems.

Theorem 2.1 *Let $\mathcal{X}$ be an arbitrary graph property in* {Odd-Hole-Free, Even-Hole-Free, Perfect}*. Then $\mathcal{X}$ is polynomially time recognizable if and only if $\mathcal{X}$ is PPS-Hereditary.*

Theorem 2.2 *Let $\mathcal{X}$ be a graph class. Then $\mathcal{X}$ is PPS-Hereditary if and only if the following three statements hold.*

1. *$\mathcal{X}$ is hereditary.*
2. *There exists a positive real number t_1 such that we can make a unified algorithm adaptable to every graph $G \in \mathcal{X}$ which calculates the following problem in $O(|V(G)|^{t_1})$-time.*

   INSTANCE: A graph $G \in \mathcal{X}$ and the path parities of its all non-adjacent vertex-pairs.

   TASK: For every vertex $v \in V(G)$, determine the path parities of the all non-adjacent vertex-pairs in $G - v$.

3. *There exists a positive real number t_2 such that we can make a unified algorithm adaptable to every graph $G \in \mathcal{X}$ which calculates the following problem in $O(|V(G)|^{t_2})$-time.*

   INSTANCE: A graph $G \in \mathcal{X}$, the path parities of all non-adjacent vertex-pairs of G, such a non-adjacent vertex-pair (p, q) of G that, for a vertex $w \notin V(G)$, the graph $H(p, q) := (V(G) \cup \{w\}, E(G) \cup \{(w, p), (w, q)\})$ is also in $\mathcal{X}$.

   TASK: Determine the path parities of all non-adjacent vertex-pairs in $H(p, q)$.

References

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