Counting simplicial decompositions of surfaces with boundaries

Olivier Bernardi *and Juanjo Rué †

It is well-known that the triangulations of the disc with $n+2$ vertices on its boundary are counted by the *nth Catalan number* $C(n) = \frac{1}{n+1}\binom{2n}{n}$. This work deals with the generalisation of this problem to any arbitrary compact surface $\mathbb{S}$ with boundary. In particular, we will be interested in the asymptotic number of simplicial decompositions of the surface $\mathbb{S}$ having n vertices, all of them lying on the boundary.

A *map* on a surface $\mathbb{S}$ is a decomposition of $\mathbb{S}$ into a finite number of 0-cells or *vertices*, 1-cells or *edges* and 2-cells or *faces*. The *degree* of a face is the number of incident edges *counted with multiplicity* (an edge is counted twice if both sides are incident to the face). A map is *triangular* if every face has degree 3. More generally, given a set $\Delta \subset \{1, 2, 3, \ldots\}$, a map is Δ-angular if the degree of any face belongs to Δ. A map is a *cellular dissection* (or shortly, a *dissection*) if any face of degree k is incident with k distinct vertices and the intersection of any two faces is either empty, a vertex or an edge. It is easy to see that triangular maps are dissections if and only if they have no loops nor multiple edges. These maps are called *simplicial decompositions* of $\mathbb{S}$.

In this work we enumerate asymptotically simplicial decompositions of an arbitrary surface $\mathbb{S}$ with boundaries. More precisely, we consider the set $\overrightarrow{\mathcal{D}}_{\mathbb{S}}(n)$ of rooted simplicial decompositions of $\mathbb{S}$ having n vertices, all of them lying on the boundary, and prove the asymptotic behaviour

$$|\overrightarrow{\mathcal{D}}_{\mathbb{S}}(n)| \sim_{n\to\infty} c\,(\mathbb{S})\, n^{-3/2\chi(\mathbb{S})} 4^n, \tag{1}$$

where $\chi(\mathbb{S})$ is the *Euler characteristic of* $\mathbb{S}$ and $c\,(\mathbb{S})$ is a constant that only depends on $\mathbb{S}$. For instance, the disc $\mathbb{D}$ has Euler characteristic $\chi = 1$ and the number of simplicial decompositions is $C(n-2) \sim \frac{1}{16\sqrt{\pi}} n^{-3/2} 4^n$.

We also study limit laws over these structures. We call *non-structuring* the edges which either belong to the boundary of $\mathbb{S}$ or separate the surface into two parts, one of which is isomorphic to the disc (the other being isomorphic to $\mathbb{S}$); the other edges (in particular those that join distinct boundaries) are called *structuring*. We determine the limit law for the number of structuring edges in simplicial decompositions.

*CNRS. Département de Mathématiques. Université Paris-Sud, 91045 Orsay Cedex, France. E-mail: olivier.bernardi@math.u-psud.fr

†Departament de Matemàtica Aplicada 2, Universitat Politècnica de Catalunya, Barcelona, Spain. E-mail: juan.jose.rue@upc.edu

More concretely, we show that the number $U_n(\mathcal{D}_{\mathbb{S}})$ of structuring edges in uniformly random simplicial decompositions of a surface $\mathbb{S}$ with n vertices rescalled by a factor $n^{-1/2}$ converges in distribution toward a random variable $U_{\chi(\mathbb{S})}$ with probability density function

$$g_{\chi(\mathbb{S})}(t) = \frac{2}{\Gamma\left(\frac{1-3\chi(\mathbb{S})}{2}\right)} \left(\frac{t}{2}\right)^{3\chi(\mathbb{S})} e^{-t^2/4}\mathbb{I}_{[0,\infty)}(t), \tag{2}$$

which only depends on the Euler characteristic of $\mathbb{S}$.

We also generalise the enumeration and limit law results to Δ-angular dissections for any set of degree $\Delta \subset \{3, 4, 5, \ldots\}$.

Our results are obtained by exploiting a decomposition of the different families of maps which is reminiscent of Wright's work on graphs with fixed excess [6] or, more recently, of work by Chapuy, Marcus and Schaeffer on the enumeration of unicellular maps [2]. This decomposition easily translates into an equation satisfied by the corresponding generating function. We then apply classical enumeration techniques based on the analysis of the generating function singularities [3]. Limit laws results can also be obtained by applying the so-called *method of moments* (see [1]).

This work generalises some of the results given in [4] and [5] where the asymptotic number of simplicial decompositions of the cylinder and the Möbius band are obtained. As in these papers, we will only be dealing with maps having all their vertices on the boundary of the surface. This is a sharp restriction which contrasts with most papers in map enumeration.

References

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