# A necessary and sufficient condition for a spanning tree with specified vertices having large degrees

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Let $G$ be a connected graph and let $X$ be a subset of $V(G)$. Let $f$ be a mapping from $X$ to $\{2, 3, \ldots\}$. In this talk, we concentrate on a spanning tree $T$ in $G$ such that $d_T(x) \geq f(x)$ for any $x \in X$, where $d_T(x)$ is the degree of $x$ in $T$. For an easy notation, we call such a spanning tree *an* $(X, f)$*-tree*.

In particular, we will consider a necessary and sufficient condition for the existence of an $(X, f)$-tree. But, in general, it seems to be difficult to give it, because the problem of determining whether a graph $G$ has an $(X, f)$-tree or not is at least as hard as the problem for a hamilton path, which is well-known as an $NP$-complete problem. Formally, when we would like to determine whether a given graph $G$ has a hamilton path, we construct a graph $G'$ by joining two new vertices to $G$, and let $X := V(G)$ and $f \equiv 2$. Then $G$ has a hamilton path if and only if $G'$ has an $(X, f)$-tree.

However, when we restrict $X$ to an independent set, we can obtain a necessary and sufficient condition for the existence of an $(X, f)$-tree. Frank and Gyárfás, and independently, Kaneko and Yoshimoto gave the following result.

**Theorem 1.** (Frank and Gyárfás [1], Kaneko and Yoshimoto [2]) *Let $G$ be a connected graph, let $X \subseteq V(G)$ be an independent set, and let $f$ be a mapping from $X$ to $\{2, 3, \ldots\}$. Then there exists an $(X, f)$-tree in $G$ if and only if for any nonempty subset $S \subseteq X$,*

$$|\bigcup_{x \in S} N_G(x)| \geq \sum_{x \in S} f(x) - |S| + 1.$$

In the case where $X$ is not an independent set, we similarly obtain the following necessary condition for the existence of an $(X, f)$-tree. For a vertex set $S \subseteq V(G)$, we denote the number of components of $G[S]$ by $\omega_G(S)$, where $G[S]$ is the subgraph of $G$ induced by $S$.

**Proposition 2.** *Let $G$ be a connected graph, let $X \subseteq V(G)$ and let $f$ be a mapping from $X$ to $\{2, 3, \ldots\}$. If there exists an $(X, f)$-tree in $G$, then for any nonempty*

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*subset* $S \subseteq X$,

$$(1) \qquad |\bigcup_{x\in S} N_G(x) - S| \geq \sum_{x\in S} f(x) - 2|S| + \omega_G(S) + 1.$$

Note that for an independent set $X$, the condition (1) is equivalent to the necessary condition in Theorem 1, because $\bigcup_{x\in S} N_G(x) - S = \bigcup_{x\in S} N_G(x)$ and $\omega_G(S) = |S|$.

One might expect that the condition (1) is also a sufficient condition. However, as mentioned before, it is impossible to show that (unless $P = NP$). In fact, even when $X$ induces a path consisting of four vertices, there exists an example that satisfies the condition (1) but has no $(X, f)$-trees.

By the gap between the case of an independent set and that of a general vertex set, it is natural to ask why the gap appears, in other words, what properties guarantee the condition (1) is also a sufficient condition. The main purpose of this talk is to give an answer of this question by showing the following result; when $G[X]$ has no induced path of order four, the condition (1) is also a sufficient condition.

**THEOREM 3.** *Let $G$ be a connected graph, and let $X \subseteq V(G)$ such that each component of $G[X]$ has no induced path of order four. Let $f$ be a mapping from $X$ to $\{2, 3, \ldots\}$. Then there exists an $(X, f)$-tree in $G$ if and only if for any nonempty subset $S \subseteq X$,*

$$|\bigcup_{x\in S} N_G(x) - S| \geq \sum_{x\in S} f(x) - 2|S| + \omega_G(S) + 1.$$

## References


[1] A. Frank and A. Gyárfás, How to orient the edges of a graph? *Colloq. Math. Soc. Jànos Bolyai* **18** (1976) 353–364.

[2] A. Kaneko and K. Yoshimoto, On spanning trees with restricted degrees, *Inform. Process. Lett.* **73** (2000) 163–165.