Forests in graphs

Katsuhiro Ota*

Many problems in graph theory, especially in extremal graph theory, can be stated as the one finding a common subgraph of graphs in a given family of graphs. More formally, for a family $\mathcal{G}$ of graphs, define *the set of common subgraphs* of the family $\mathcal{G}$ as

$$\mathrm{CS}(\mathcal{G}) \;=\; \{H \mid H \subseteq G \text{ for all } G \in \mathcal{G}, \text{ and } H \text{ contains no isolated vertex}\}.$$

For example, Dirac's theorem on hamiltonian cycle states that the cycle of length n is contained in $\mathrm{CS}(\mathcal{G}(n, n/2))$, where $\mathcal{G}(n, k)$ denotes the set of graphs of order n with minimum degree at least k. Turan's theorem [3] can be stated that $K_r \in \mathrm{CS}(\mathcal{G}_e(n, t_{r-1}(n) + 1))$, where $t_{r-1}(n)$ stands for the Turan number, and $\mathcal{G}_e(n, m)$ is the set of graphs with n vertices and m edges. More generally, Erdős and Stone's theorem [2] states that any graph H with chromatic number r is in $\mathrm{CS}(\mathcal{G}_e(n, (1 + o(1))t_{r-1}(n)))$ if $n \gg |V(H)|$.

Let us consider $\mathcal{G}(n, k)$ the set of graphs with a given *constant* minimum degree k. Define

$$\mathcal{G}(\geq n_0, k) \;=\; \bigcup_{n \geq n_0} \mathcal{G}(n, k),$$

the set of all graphs of order at least n_0 with minimum degree at least k. We write $\mathrm{CS}(\geq n_0, k)$ for $\mathrm{CS}(\mathcal{G}(\geq n_0, k))$.

Since there exists a graph with arbitrarily large girth and minimum degree, we can see that every graph in $\mathrm{CS}(\geq n_0, k)$ is acyclic. Thus, in order to determine the set $\mathrm{CS}(\geq n_0, k)$, we have to consider the following function $\eta(F)$ for each forest F:

$$\eta(F) \;=\; \min\{k \mid \exists n_0,\ F \in \mathrm{CS}(\geq n_0, k)\}.$$

It is not difficult to see that if T is a tree then $\eta(T) = |V(T)| - 1$. More generally, Brandt [1] proved that if F is a forest then $\eta(F) \leq |E(F)|$. More precisely, it is proved that if G is a graph with minimum degree at least $|E(F)|$ and $|V(G)| \geq |V(F)|$, then F is a subgraph of G. However, this bound is not tight in general. For example, if $F = K_{1,k} \cup K_{1,k-1} \cup \cdots \cup K_{1,1}$, then $\eta(F) = k$ while $|E(F)| = k(k-1)/2$.

For a forest F and a positive integer t, we define $s(F, t)$ to be the minimum cardinality of an independent set $S \subset V(F)$ such that $F - S$ has no component of order greater than t. It is not difficult to prove the following proposition.

*Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan. E-mail: ohta@math.keio.ac.jp

PROPOSITION 1. $\eta(F) \geq \max\{s(F,t)+t-1 \mid t \geq 1, s(F,t) \geq 1\}$.

Actually, we conjecture that the equality always holds in this proposition. We have proved that the equality holds when

- F is a tree,
- each component of F is a path,
- each component of F is a star, or
- one component of F is a path and the others are stars.

Obviously, the set $\mathrm{CS}(\mathcal{G})$ is closed under the operation taking a subgraph (without isolated vertices). Thus, it is most important to determine all maximal elements in $\mathrm{CS}(\mathcal{G})$ (maximal with respect to the subgraph/supergraph relation). Let $\mathrm{MCS}(\mathcal{G})$ denote the set of all maximal elements in $\mathrm{CS}(\mathcal{G})$, i.e.,

$$\mathrm{MCS}(\mathcal{G}) \;=\; \{H \in \mathrm{CS}(\mathcal{G}) \mid H \subseteq H' \in \mathrm{CS}(\mathcal{G}) \text{ implies } H = H'\}$$

In particular, we consider $\mathrm{MCS}(\geq n_0, k) = \mathrm{MCS}(\mathcal{G}(\geq n_0, k))$ for a fixed k. For a small k, we obtained the following.

PROPOSITION 2.

(1) *For any integer* $n_0 \geq 5$, $\mathrm{MCS}(\geq n_0, 2) = \{P_3 \cup P_2\}$.
(2) *For any integer* $n_0 \geq 9$, $\mathrm{MCS}(\geq n_0, 3) = \{P_4 \cup P_3, K_{1,3} \cup K_{1,2} \cup K_{1,1}\}$.
(3) *For any sufficiently large* n_0, $\mathrm{MCS}(\geq n_0, 4)$ *exactly consists of the following six forests, where* T_5 *is the unique tree of degree sequence* $3,2,1,1,1$.

$$P_5 \cup P_4,\ T_5 \cup P_4,\ P_5 \cup K_{1,3} \cup K_{1,2},\ T_5 \cup K_{1,3} \cup K_{1,2},$$
$$K_{1,4} \cup P_4 \cup K_{1,2},\ K_{1,4} \cup K_{1,3} \cup K_{1,2} \cup K_{1,1}.$$

Also, we found some forests in $\mathrm{MCS}(\geq n_0, k)$ for each integer $k \geq 2$.

PROPOSITION 3. *For each integer* $k \geq 2$, *the following forests are in* $\mathrm{MCS}(\geq n_0, k)$ *for sufficiently large* n_0.

- $P_{k+1} \cup P_k$,
- $K_{1,k} \cup K_{1,k-1} \cup \cdots \cup K_{1,1}$,
- $P_{k+1} \cup K_{1,k-1} \cup \cdots \cup K_{1,\lceil k/2 \rceil}$, $K_{1,k} \cup P_k \cup K_{1,k-2} \cup \cdots \cup K_{1,\lceil (k-1)/2 \rceil}$, *etc.*

References

[1] S. Brandt, Subtrees and subforests of graphs, J. Combin. Theory B 61 (1994), 63–70.

[2] P. Erdős and A.H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946), 1087–1091.

[3] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, Mat. Fiz. Lapok 48 (1941), 436–451.