On partitional labelings of graphs

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In [5], Graham and Sloane defined a graph G of order p and size q with $q \geq p$ to be *harmonious* if there exists an injective function $f : V(G) \to \mathbf{Z}_q$ such that when each edge $uv \in E(G)$ is labeled $f(u) + f(v) \pmod{q}$, the resulting edge labels are distinct. Such a function is called a *harmonious labeling*. If G is tree so that $q = p - 1$, exactly two vertices are labeled the same; otherwise, the definition is the same.

In [4], Grace defined a *sequential labeling* of a graph G of size q as an injective function $f : V(G) \to [0, q-1]$ such that when each edge $uv \in E(G)$ is labeled $f(u) + f(v)$, the resulting edge labels are $[m, m+q-1]$ for some positive integer m. A graph is called *sequential* if it admits a sequential labeling.

We will denote the set of integers $\{m, m+1, \ldots, n\}$ by simply writing $[m, n]$.

The n-dimensional cube Q_n serves as useful models for a broad range of applications such as circuit design, communication network addressing, parallel computation and computer architecture; hence, we concern in this section the sequential labeling of Q_n.

Theorem 1. *Let n be an integer with $n \geq 2$. Then the n-dimensional cube Q_n is sequential if and only if $n \neq 2, 3$.*

We introduce a new type of sequential labeling. A sequential labeling f of a graph G with $2t + s$ edges is called a *partitional labeling* if G is bipartite with two partite sets X and Y of the same cardinality s such that $f(x) \leq t + s - 1$ for all $x \in X$ and $f(y) \geq t - s$ for all $y \in Y$, and there is a positive integer m such that the induced edge labels are partitioned into three sets $[m, m+t-1]$, $[m+t, m+t+s-1]$ and $[m+t+s, m+2t+s-1]$ with the property that there is an involution π which is an automorphism of G such that π exchanges X and Y, $x\pi(x) \in E(G)$ for all $x \in X$, and $\{f(x) + f(\pi(x)) \,|\, x \in X\} = [m+t, m+t+s-1]$. A graph is called *partitional* if it admits a partitional labeling. With this definition in hand, we are now able to present the following result.

Theorem 2. *If G is partitional, then $G \times K_2$ is partitional.*

If we apply Theorem 2 repeatedly, then we obtain the following result.

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Theorem 3. *If G is partitional, then $G \times Q_n$ is partitional for every nonnegative integer n.*

In light of Theorem 3 and the fact that every partitional labeling is sequential, harmonious and felicitous (see [3] for the definition of a felicitous labeling), we have the following result.

Corollary 4. *If G is partitional, then $G \times Q_n$ is sequential, harmonious and felicitous for every nonnegative integer n.*

Applying Theorem 3 and Corollary 4 with $G \cong Q_4$, we obtain the following result, which is a refinement of the 'if' part of Theorem 1.

Theorem 5. *Let n be an integer with $n \geq 4$. Then the n-dimensional cube Q_n is partitional, sequential, harmonious and felicitous.*

In [2], Gallian and Jungreis have shown that the book $S_{2m} \times Q_1$ is sequential for each positive integer m. In fact, their sequential labeling of $S_{2m} \times Q_1$ satisfy the conditions to be partitional. Thus, we are able to state the following result.

Theorem 6. *For every positive integer m, the book $S_{2m} \times Q_1$ is partitional.*

In [1], the ladder $P_{2m+1} \times Q_1$ has been shown to be super edge-magic for every positive integer m, and it induces the partitional labeling of $P_{2m+1} \times Q_1$ by subtracting 1 from its vertex labels; hence, we have the following result.

Theorem 7. *For every positive integer m, the ladder $P_{2m+1} \times Q_1$ is partitional.*

Corollary 8. *For any two positive integers m and n, the generalized book $S_{2m} \times Q_n$ is partitional, sequential, harmonious and felicitous.*

Corollary 9. *For every two positive integers m and n, the generalized ladder $P_{2m+1} \times Q_n$ is partitional, sequential, harmonious and felicitous.*

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