Distinguishing numbers and faithfulness of embedding of graphs on surfaces

Seiya Negami*

The distinguishing number of a graph G is the minimum number of distinct labels assigned to its vertices so as to break its symmetry, and is defined formally as follows. An assignment $c : V(G) \to \{1, \dots, d\}$ of d distinct labels to the vertices of G is called a *d-distinguishing labeling* if no automorphism of G other than the identity map preseves the labels given by c. A graph G is said to be *d-distinguishable* if G admits a d-distinguishing labeling. The *distinguishing number* of G is defined as the minimum number d such that G is d-distinguishable and is denoted by $D(G)$. For example, $D(K_n) = n$ and $D(K_{n,n,n}) = n + 1$ for $n \geq 2$.

Although the notion of distinguishing number is defined for abstract graphs, there have been studies about it with topological aspect, as listed in the references. For example, Fukuda, Negami and Tucker [2] have proved the following theorem, where $S_k H \cong H + \overline{K_k}$ stands for the join of a graph H and k isolated points, called the *suspension* of H with k vertices:

Theorem 1. (Fukuda, Negami and Tucker [2]) *Every 3-connected planar graph is 2-distinguishable, except K_4, $K_{2,2,2}$, W_4, W_5, S_2C_3, S_2C_5 and Q_3.*

It is not difficult to determine the distinguishing number of the exceptoins in this theorem:

$$D(K_4) = 4, D(K_{2,2,2}) = D(W_4) = D(W_5) = D(S_2C_3) = D(S_2C_5) = D(Q_3) = 3$$

Their proof of the above theorem is based on the fact that any 3-connected planar graph G can be embedded on the sphere "faithfully" so as to realize the symmetry of G concerning its automorphism group. In general, a graph G embedded on a closed surface F^2 is said to be *faithfully embedded* on F^2 if any automorphism of G extends to an auto-homeomorphism over F^2.

As preparation for further studies under more general situation, the author [5] has establisthed the following theorem, extending the formulation of distinguishing number. A 3-connected graph G is said to be *polyhedral* on a closed surface F^2 if each face of G is a 2-cell bounded by a cycle and if the intersection of any two faces is either empty, a single vertex or a single edge with its ends. Consider a subgroup Γ in $\mathrm{Aut}(G)$. We define the distinguishing number and the faithfulness of embedding

*Faculty of Education and Human Sciences, Yokohama National University, 79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501, Japan. E-mail: negami@edhs.ynu.ac.jp

for the pair (G, Γ) in the same way as above, restricting automorphisms to ones belonging to Γ:

Theorem 2. *Let G be a polyhedral graph embedded on a closed surface F^2 and Γ a subgroup in $\mathrm{Aut}(G)$. If (G, Γ) is faithfully embedded on F^2, then either (G, Γ) is 2-distinguishable, or G is isomorphic to one of the followings:*

(i) *K_7 and $K_{3,3,3}$ on the torus and K_n with $n \geq 8$ embedded on closed surfaces as triangulations; they should be vertex-transitive and edge-transitive under Γ.*
(ii) *$C_3 \times C_3$ and $\overline{K_6 \times K_2}$; they are embedded on the torus and on the nonorientable closed surface of genus 5 as quadrangulations so as to be vertex-transitive and edge-transitive under Γ.*
(iii) *K_4, S_2C_3, S_2C_4, S_2C_5 and K_6, embedded on closed surfaces as triangulations.*
(iv) *W_4, W_5 and Q_3, embedded on the sphere as polygonal pyramids and the cube.*

Note that the pairs (G, Γ) with the exceptional graphs G in the theorem might be 2-distinguishable, depending on Γ. For example, if Γ is a non-trivial cyclic group, then we have $D(G, \Gamma) = 2$. Also, there are those graphs that cannot be faithfully embedded on any closed surface. We obtain the following corollary, excluding such graphs with $\Gamma = \mathrm{Aut}(G)$,

Corollary 3. *Every polyhedral graph faithfully embedded on a closed surface is 2-distinguishable unless $C_3 \times C_3$, K_4, S_2C_3, S_2C_4, S_2C_5, W_4, W_5 and Q_3.*

If a graph G is embedded on a closed surface F^2, but not faithfully, then there is an automorphism of G which does not extend to any auto-homeomorphism over F^2. Such an automorphism can be regarded as a re-embedding of G to F^2. Thus, the arguments on the distinguishing number of graphs not faithfully embedded on a closed surface are closely related to a re-embedding theory of graphs.

References

[1] Aishanjiang W., S. Negami and K. Yamamoto, The distinguishing number of 4-regular quadrangulations on the Klein bottle, submitted 2008.
[2] T. Fukuda, S. Negami and T. Tucker, 3-Connected planar graphs are 2-distinguishable with few exceptions, to appear in *Yokohama Math. J.* (2008).
[3] T. Fukuda and S. Negami, The distinguishing number of 4-regular quadrangulations on the torus, submitted 2008.
[4] The distinguishing number of triangulations on the projective plane, *Congressus Numernatium* **186** (2007), 135–144.
[5] The distinguishing numbers of graphs on closed surfaces, submitted 2008.
[6] T.W. Tucker, Distinguishability of maps, preprint 2005.