Domatic number of triangulations on surfaces

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This is a joint work with Ken-ichi Kawarabayashi (NII) and Tatsuya Honjo (YNU). We consider only finite simple graphs G, and we always suppose $|V(G)| = n$.

We say that $S \subset V(G)$ is *dominating* in G if for each vertex v of G, v is contained in S, or v has a neighbor in S. Since $V(G)$ is a dominating set of G, we are interested in the minimum cardinality of a dominating set of G, which is called a *dominating number* of G and denoted by $\gamma(G)$. So we have $\gamma(G) \leq n$ for any graph G, and it is easy to see that the equality holds if and only if $G = \overline{K_n}$.

Let $c : V(G) \to \{1, \ldots, k\}$, where c is not necessarily a *proper* coloring of G. We say that c is *dominating k-coloring* of G if for any $i \in \{1, \ldots, k\}$, $c^{-1}(i) = \{v \in V(G) : c(v) = i\}$ is a dominating set in G. It is easy to see that if G has a dominating k-coloring, then G also has a dominating $(k-1)$-coloring. So we are interested in the maximum value of k such that G admits a dominating k-coloring, which is called a *domatic number* of G and denoted by $\mathrm{Dom}(G)$.

By the definition of the two invariants of graphs, we have the following:

Proposition 1. *If a graph G admits a dominating k-coloring, then G has a dominating set of cardinality at most n/k. (I.e., if $\mathrm{Dom}(G) \geq k$, then $\gamma(G) \leq n/k$.)*

A *triangulation* G on a surface is a map of a simple graph with at least three vertices such that each finite face is triangular. For a triangulation on the disk, Matheson and Tarjan proved the following in order to bound a dominating number:

Theorem 2. (Matheson and Tarjan [2]) *Every triangulation on the disk has a dominating 3-coloring, that is,* $\mathrm{Dom}(G) \geq 3$.

An immediate consequence of Theorem 2 by Proposition 1 is that every triangulation on the disk has a dominating number at most $n/3$. They also proved that "$n/3$" is best possible, by showing that there exists a triangulation on the disk each of whose dominating set has cardinality $\geq n/3$. Since such an example has many vertices of degree 2 on the boundary cycle, they conjectured that every triangulation on the disk has dominating number at most $n/4$ if the boundary cycle has length exactly 3. (If this is true, then we can easily prove that the bound "$\leq n/4$" is best possible.)

Our main theorem is the following. (We have been motivated by the Plum-

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mer and Zha's result [3] dealing with dominating numbers of triangulations on the projective plane, the torus and the Klein bottle.)

Theorem 3.

(i) Let G be a triangulation on the projective plane, the torus and the Klein bottle. Then G has a domatic number at least 3.

(ii) For any surface F^2, there exists a positive integer $N(F^2)$ such that every triangulation on F^2 with representativity at least $N(F^2)$ has a domatic number at least 3.

Moreover, every surface admits a triangulation whose domatic number is exactly 3, that is, a triangulation with no dominating 4-coloring.

Corollary 4. *Every triangulation on the disc (or the sphere), the projective plane, the torus and the Klein bottle has a dominating number at most $n/3$. Moreover, the same fact holds for triangulations on any surface with sufficiently large representativity.*

The estimation of domatic numbers in Theorems 2 and 3 cannot be improved, but we conjecture that the assumption for the representativity in Theorem 3(ii) can be removed. On the other hand, the bound in Corollary 4 is not best possible, since the converse of Proposition 1 does not hold in general. Hence we conjecture the following (for which King and Pelsmajer [1] have proved it is true for plane triangulations with maximum degree 6).

Conjecture 1. *Every triangulation on any surface with sufficiently large number of vertices has a dominating number at most $n/4$.*

References

[1] E.L.C. King and M.J. Pelsmajer, Dominating Sets in Triangulations, preprint.

[2] L. Matheson and R. Tarjan, Dominating sets in planar graphs, *European J. Combin.* **17** (1996) 565–568.

[3] M.D. Plummer and X. Zha, On certain spanning subgraphs of embeddings with applications to domination, to appear in *Discrete Math.*