Polynomial time algorithms for computing Jones polynomials of certain links

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Given any link diagram, we can color the faces black and white in such a way that no two faces with a common arc are the same color. We color the unique unbounded face white. We can get the edge-labeled planar graph (see Figure 1). Its vertices are the black faces of the Tait coloring and two vertices are joined by a labeled edge if they share a crossing. The label of an edge is $+1$ or -1 according to the (conventional) rule. We call the graph the *Tait graph* of the diagram. Note that the number of the edges in the graph is equal to the number of the crossings of the diagram.

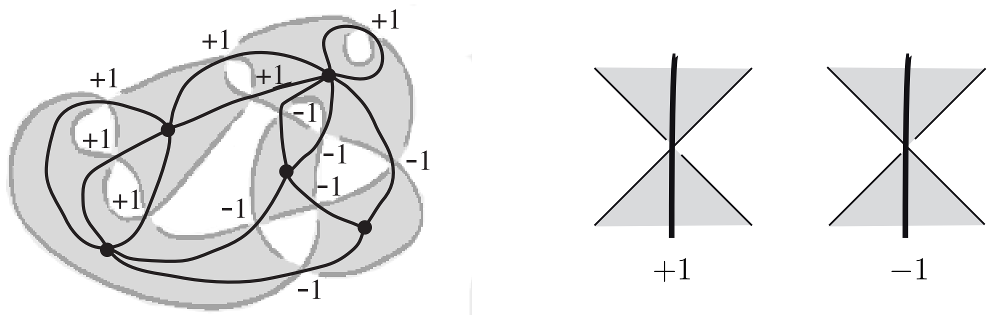

Figure 1 The Tait coloring, the Tait graph of a link diagram and labels of edges.

The Jones polynomial [3] is a useful invariant in knot theory. By using Kauffman's method [4], the Jones polynomial is computable with $\mathcal{O}(2^{\mathcal{O}(n)})$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, where n is the number of the edges in the input Tait graph. F. Jaeger, D.L. Vertigan and D.J.A. Welsh showed that computing the Jones polynomial is generally #**P**–hard [2, 10]. It is expected to require exponential time in the worst case.

Recently, it has been recognized that it is important to compute Jones polynomials for links with reasonable restrictions. J.A. Makowsky [5, 6] showed that Jones polynomials are computable in polynomial time if treewidths of input Tait graphs are bounded by a constant. J. Mighton [7] showed that Jones polynomials

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are computable with $\mathcal{O}(n^4)$ operations in polynomials of degree $\mathcal{O}(n)$ if treewidths of input Tait graphs are at most two. M. Hara, S. Tani and M. Yamamoto [1] showed that Jones polynomials of arborescent links are computable with $\mathcal{O}(n^3)$ operations in polynomials of degree $\mathcal{O}(n)$. T. Utsumi and K. Imai [9] showed that Jones polynomials of pretzel links are computable in $\mathcal{O}(n^2)$ time.

We propose algorithms for computing Jones polynomials of 2–bridge links, closed 3–braid links and Montesinos links introduced by J.M. Montesinos [8] as shown in Figure 2 with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, namely in $\mathcal{O}(n^2 \log n)$ time. 2–bridge links, closed 3–braid links and Montesinos links are basic links and have been profoundly studied in knot theory.

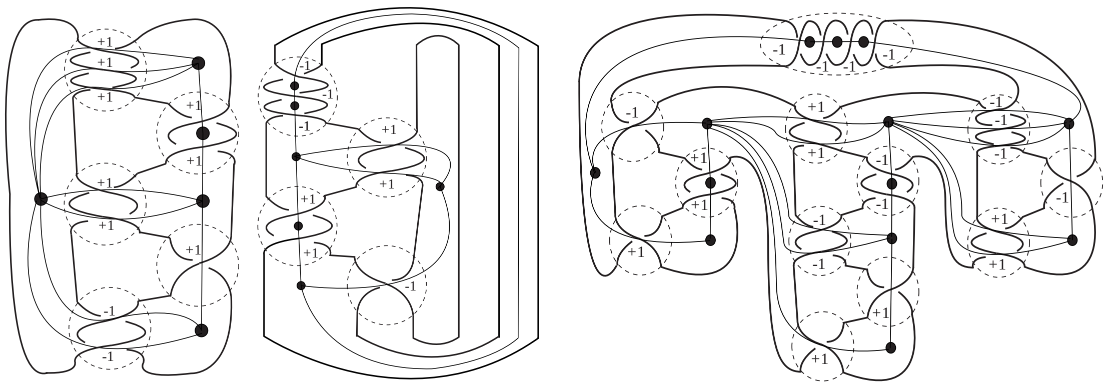

Figure 2 Tait graphs of a 2–bridge link, a closed 3–braid link and a Montesinos link.

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