K_6-Minors in triangulations on the nonorientable surface of genus 3

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It is easy to characterize graphs with no K_k-minors for any integer $k \leq 4$, as follows. For $k = 1, 2$, the problem must be trivial, and for $k = 3, 4$, those graphs are forests and *series-parallel graphs* (i.e., graphs obtained from K_3 by a sequence of replacing a vertex of degree 2 with a pair of parallel edges, or its inverse operation.) Moreover, Wagner formulated a fundamental characterization of the graphs having no K_5-minor [4]. However, a complete characterization of graphs having K_6-minor seems to be a difficult problem, in general.

In our talk, we consider the following problem: For a given triangulation G, which complete graph K_n is contained in G as a minor? It is easy to see that every triangulation G on any surface has a K_4-minor, and that every triangulation G on any non-spherical surface has a K_5-minor. The following are complete characterizations of triangulations on the projective plane and the torus with no K_6-minor:

Theorem 1. ([2])

(1) A triangulation G on the projective plane has no K_6-minor if and only if G has a K_4-quadrangulation as a subgraph.

(2) A triangulation G on the torus has no K_6-minor if and only if G has a K_5-quadrangulation as a subgraph.

Let $\mathbb{N}_k$ denote the nonorientable surface of genus k, respectively. A 4-*quadrangle* is a plane graph whose outer cycle has length 4 and all of whose inner cycles have length 3. A 4-*annulus* (A, C_1, C_2) is a triangulation on the annulus with boundary cycles C_1, C_2 such that $|C_1| = |C_2| = 4$, where we allow $C_1 \cap C_2 \neq \emptyset$. We say that (A, C_1, C_2) is *nested* if there are m (≥ 2) homotopic 4-cycles $D_1, \ldots, D_m$ lying on the annulus in this order such that $C_1 = D_1, C_2 = D_m$, and $V(D_j) \cap V(D_{j+1}) \neq \emptyset$ for each j. Let H be a K_4-quadrangulation on the projective plane with faces F_1, F_2, F_3. A *Möbius quadrangle* is a map on the Möbius band obtained from a H by

(i) removing the interior of F_i, for $i = 1, 2, 3$,

(ii) for $i = 1, 2$, pasting a 4-quadrangle Q to the boundary of F_i, and

(iii) pasting a nested 4-annulus or a 4-annulus with an essential 3-cycle to F_3 so that one of its two boundary components and the boundary of F_3 are identified.

Similarly, we define a *torus quadrangle* from a K_5-quadrangulation on the torus.

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The following are the results for the Klein bottle and the double torus.

THEOREM 2. ([1, 3])

(1) A triangulation G on the Klein bottle has no K_6-minor if and only if G is obtained from two Möbius quadrangles by identifying their boundaries.

(2) A triangulation G on the double torus has no K_6-minor if and only if G is obtained from two torus quadrangles by identifying their boundaries.

In this talk, we consider the nonorientable surface of genus 3 and prove the following.

THEOREM 3. *A triangulation G on $\mathbb{N}_3$ has no K_6-minor if and only if G is obtained by one of the following procedures.*

(i) Let H be a K_4-quadrangulation with faces F_1, F_2, F_3, and replace the interior of F_1, F_2 with Möbius quadrangles respectively, and replace F_3 with a 4-quadrangle,

(ii) Let P be a plane graph all of whose faces are triangular, except exactly three quadrilateral faces A, B, C with $\partial A = a_1a_2a_3a_4$, $\partial B = b_1b_2b_3b_4$, $\partial C = c_1c_2c_3c_4$, such that either $a_1 = b_1 = c_1$ or $a_3 = b_1$, $b_3 = c_1$, $c_3 = a_1$. Replace the interior of quadrilateral faces A, B, C by a Möbius quadrangle.

(iii) Paste a Möbius quadrangle and a torus quadrangle along their boundaries.

We have the following, since all triangulations on $\mathbb{N}_3$ with no K_6-minor has a separating essential cycle of length at most 4 and an essential non-separating 3-cycle, by Theorem 3.

COROLLARY 4. *Every 5-connected triangulation on $\mathbb{N}_3$ has a K_6-minor, and every 5-representative triangulation on $\mathbb{N}_3$ has a K_6-minor.*

Corollary 4 holds for all the surfaces we dealt so far, and hence we conjecture that this holds for all surfaces.

References

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