On irreducible rectangle tilings without a spiral

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Let R^2 be the Euclidean plane. A subset r of R^2 is a *upright rectangle* if $r = [a,b] \times [c,d]$ for some real numbers $a < b$ and $c < d$ where $[p,q]$ denotes the closed interval with end points p and q. Then the boundary ∂r of r is defined by $\partial r = \{a,b\} \times [c,d] \cup [a,b] \times \{c,d\}$. The interior int$r$ of r is defined by int$r = r - \partial r$. The points $(a,c),(a,d),(b,c),(b,d)$ are called the *corners* of r. Each of $\{a\} \times [c,d]$, $\{b\} \times [c,d]$, $[a,b] \times \{c\}$ and $[a,b] \times \{d\}$ is called an *edge* of r. All rectangles in this article are upright and in the following we omit the adjective upright. Throughout this article S denotes a rectangle. Let $\mathcal{R}$ be a set of rectangles. We say that $\mathcal{R}$ is a *tiling of* S if the following conditions hold:

(1) $S = \bigcup_{r \in \mathcal{R}} r$,

(2) if $r, s \in \mathcal{R}$ and $r \neq s$ then $r \cap s = \partial r \cap \partial s$,

(3) for any $x \in S$ there is a neighbourhood N of x in S such that only finitely many elements of $\mathcal{R}$ have non-empty intersection with N.

Then we call each element in $\mathcal{R}$ a *tile*. If $\mathcal{R}$ is a tiling of S then, by the compactness of S, we have that $\mathcal{R}$ is a finite set.

By a *line* we mean a straight line in R^2. By a *half line* we mean a straight half line in R^2. By a *line segment* we mean a straight line segment in R^2. Note that a line has no end points and separates R^2 into two regions. A half line has just one end point and a line segment has just two end points. Note that if a line L is oriented then the left-hand side of L and the right-hand side of L are well-defined.

Let $\mathcal{R}$ be a tiling of S, r a tile in $\mathcal{R}$ and x a corner of r. Let $\deg(x, \mathcal{R})$ be the number of tiles in $\mathcal{R}$ containing x. Then we have that $\deg(x, \mathcal{R}) = 1, 2, 3$ or 4 and the first two cases occur in the case that $x \in \partial S$. We call $\deg(x, \mathcal{R})$ the *degree* of x in $\mathcal{R}$. Suppose that x is a corner of r and $\deg(x, \mathcal{R}) = 3$. Let s and t be the other tiles in $\mathcal{R}$ containing x. Let L be the line containing the line segment $s \cap t$. We orient L by the direction from x to the other end point of $s \cap t$. We say that a corner x of r with $\deg(x, \mathcal{R}) = 3$ is a *right* (resp. *left*) *type corner of* r if r is contained in the left-hand (resp. right-hand) side of L. We say that a tile r in $\mathcal{R}$ is a *right* (resp. *left*) *spiral* if every corner of r is a right (resp. left) type corner. We say that a tile r in $\mathcal{R}$ is a *spiral* if it is a right spiral or a left spiral.

We say that a tile r in $\mathcal{R}$ is a *right* (resp. *left*) *pseudo spiral* if r is a right (resp.

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left) spiral, or one of the corners of r has degree four and the other corners are right (resp. left) type corners of r. We say that a tile r in $\mathcal{R}$ is a *pseudo spiral* if it is a right pseudo spiral or a left pseudo spiral.

We say that a tiling $\mathcal{R}$ of S is *irreducible* if for any tiles r and s in $\mathcal{R}$ with $r \neq s$ the union $r \cup s$ is not a rectangle. We say that a tiling $\mathcal{R}$ of S is *strongly irreducible* if it is irreducible, and for any proper subset $\mathcal{Q}$ of $\mathcal{R}$ which contains at least two tiles the union $\bigcup_{r \in \mathcal{Q}} r$ is not a rectangle.

A tiling $\mathcal{R}$ is *nontrivial* if $\mathcal{R}$ contains at least two tiles. A tiling $\mathcal{R}$ is *generic* if every corner of every tile in $\mathcal{R}$ has degree less than 4.

Theorem 1. ([1]) *Let $\mathcal{R}$ be a nontrivial irreducible tiling of a rectangle. Then $\mathcal{R}$ contains a pseudo spiral.*

By definition a pseudo spiral in a generic tiling is a spiral. Therefore we immediately have the following corollary.

Corollary 2. ([1]) *Let $\mathcal{R}$ be a nontrivial generic irreducible tiling of a rectangle. Then $\mathcal{R}$ contains a spiral.*

We consider the case that a tiling does not contain a spiral, and obtain the following theorem.

Theorem 3. *Let $\mathcal{R}$ be a nontrivial irreducible tiling of a rectangle. If $\mathcal{R}$ does not contain a spiral, then $\mathcal{R}$ contains at least four pseudo spirals.*

References

[1] Tomoe Motohashi,Kouki Taniyama, An irreducible rectangle tiling contains a spiral, submitted 2008. Journal of Geometry,