Crossing numbers of graphs on the plane and on other surfaces

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(joint work with S. Cabello, M. DeVos, and R. Šamal)

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Abstract

In this survey talk we shall overview some recent results about the crossing number of graphs. The planar crossing number as well as crossing numbers for other surfaces will be considered. Two of the subjects are presented in more details in this note.

1 Introduction

Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. Besides its mathematical interest, there are numerous applications, most notably those in VLSI design [2, 8, 9], in combinatorial geometry and even in number theory, see, e.g, [17]. We refer to [10, 15] and to [18] for more details about diverse applications of this important notion.

A *drawing* of a graph G is a representation of G on some surface, usually on the Euclidean plane $\mathbb{R}^2$, where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their endvertices. A drawing is *clean* if the interior of every arc representing an edge contains no points representing the vertices of G. If interiors of two arcs intersect or if an arc contains a vertex of G in its interior we speak about crossings of the drawing. More precisely, a *crossing* of a drawing $\mathcal{D}$ is a pair $(\{e, f\}, p)$, where e and f are distinct edges and $p \in \mathbb{R}^2$ is a point that belongs to interiors of both arcs representing e and f in $\mathcal{D}$. If the drawing is not clean, then the arc of an edge e may contain in its interior a point $p \in \mathbb{R}^2$ that represents a vertex v of G. In such a case, the pair $(\{v, e\}, p)$ is also referred to as a *crossing* of $\mathcal{D}$.

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The number of crossings of $\mathcal{D}$ is denoted by $\mathtt{cr}(\mathcal{D})$ and is called the crossing number of the drawing $\mathcal{D}$. The *crossing number* $\mathtt{cr}(G)$ of a graph G is the minimum $\mathtt{cr}(\mathcal{D})$ taken over all clean drawings $\mathcal{D}$ of G in the plane. When each edge e of G has a weight $w_e \in \mathbb{N}$, the weighted crossing number $\mathtt{wcr}(\mathcal{D})$ of a clean drawing $\mathcal{D}$ is the sum $\sum w_e \cdot w_f$ over all crossings $(\{e,f\},p)$ in $\mathcal{D}$. The *weighted crossing number* $\mathtt{wcr}(G)$ of G is the minimum $\mathtt{wcr}(\mathcal{D})$ taken over all clean drawings $\mathcal{D}$ of G. Of course, if all edge-weights are equal to 1, then $\mathtt{wcr}(G) = \mathtt{cr}(G)$.

We can define the *crossing number* of a graph for any given surface S. In that case, we consider clean drawings $\mathcal{D}$ in S and define $\mathtt{cr}_S(G)$ as the minimum $\mathtt{cr}(\mathcal{D})$ taken over all clean drawings $\mathcal{D}$ of G in S.

2 Near-planar graphs

A nonplanar graph G is *near-planar* if it contains an edge e such that $G - e$ is planar. Such an edge e is called a *planarizing edge*. It is easy to see that near-planar graphs can have arbitrarily large crossing number. However, it seems that computing the crossing number of near-planar graphs should be much easier than in unrestricted cases. This is supported by a less known, but particularly interesting result of Riskin [13], who proved that the crossing number of a 3-connected cubic near-planar graph G can be computed easily as the length of a shortest path in the geometric dual graph of the planar subgraph $G - x - y$, where $xy \in E(G)$ is the edge whose removal yields a planar graph. Riskin asked if a similar correspondence holds in more general situations, but this was disproved by Mohar [12] (see also [6]). Another relevant paper about crossing numbers of near-planar graphs was published by Hliněný and Salazar [7].

Several generalizations of Riskin's result are indeed possible. Cabello and Mohar [12, 4] provided efficiently computable upper and lower bounds on the crossing number of near-planar graphs in a form of min-max relations. These relations can be extended to the non-3-connected case and even to the case of weighted edges. As far as we know, these results are the first of their kind in the study of crossing numbers. It is shown that they generalize and improve some known results and we foresee that generalizations and further applications are possible.

On the other hand, Cabello and Mohar [4] showed that computing the crossing number of weighted near-planar graphs is NP-hard. This discovery is a surprise and brings more questions than answers.

Let G_0 be a plane graph and let x, y be two of its vertices. A simple (polygonal) arc $\gamma : [0,1] \to \mathbb{R}^2$ is an (x,y)-*arc* if $\gamma(0) = x$ and $\gamma(1) = y$. If $\gamma(t)$ is not a vertex of G_0 for every t, $0 < t < 1$, then we say that γ is *clean*. For an (x,y)-arc γ we define the crossing number of γ with G_0 as

$$\mathtt{cr}(\gamma, G_0) = |\{t \mid \gamma(t) \in G_0 \text{ and } 0 < t < 1\}|. \qquad (1)$$

This definition extends to the weighted case as follows. If the graph G_0 is weighted and the edge xy realized by an (x,y)-arc γ also has weight w_{xy}, then each crossing of γ with an edge e contributes $w_{xy} \cdot w_e$ towards the value $\mathtt{cr}(\gamma, G_0)$, and each crossing $(\{v, xy\}, p)$ of xy with a vertex of G_0 contributes 1 (independently of the edge-weights).

Using this notation, we define the *dual distance*

$$d^*(x, y) = \min\{\mathtt{cr}(\gamma, G_0) \mid \gamma \text{ is a clean } (x, y)\text{-arc}\}.$$

We also introduce a similar quantity, the *facial distance* between x and y:

$$d'(x, y) = \min\{\mathtt{cr}(\gamma, G_0) \mid \gamma \text{ is an } (x, y)\text{-arc}\}.$$

It should be observed at this point that the value $d'(x, y)$ is independent of the weights – since all weights are integers, we can replace each crossing of an edge with a crossing through an incident vertex and henceforth replace weight contributions simply by counting the number of crossings.

Let $G^*_{x,y}$ be the geometric dual graph of $G_0 - x - y$. Then $d^*(x, y)$ is equal to the distance in $G^*_{x,y}$ between the two vertices corresponding to the faces of $G_0 - x - y$ containing x and y. Of course, in the weighted case the distances are determined by the weights of their dual edges. This shows that $d^*(x, y)$ can be computed in linear time by using known shortest path algorithms for planar graphs. Similarly, one can compute $d'(x, y)$ in linear time by using the vertex-face incidence graph (see [11]).

Clearly, $d'(x, y) \leq d^*(x, y)$. Note that d^* and d' depend on the embedding of G_0 in the plane. However, if G_0 is (a subdivision of) a 3-connected graph, then this dependency disappears since G_0 has essentially a unique embedding. To compensate for this dependence, we define $d^*_0(x, y)$ (and $d'_0(x, y)$) as the minimum of $d^*(x, y)$ (resp. $d'(x, y)$) taken over all embeddings of G_0 in the plane.

The following proposition is clear from the definition of d^*:

Proposition 2.1 *If G_0 is a weighted planar graph and $x, y \in V(G_0)$, then $\mathtt{cr}(G_0 + xy) \leq d^*_0(x, y)$.*

This result shows that the value $d^*_0(x, y)$ is of interest. Gutwenger, Mutzel, and Weiskircher [6] provided a linear-time algorithm to compute $d^*_0(x, y)$. Cabello and Mohar [12, 4] study $d^*_0(x, y)$ from a combinatorial point of view and obtain a min-max characterization that results very useful.

Riskin [13] proved the following strengthening of Proposition 2.1 in a special case when G_0 is 3-connected and cubic:

Theorem 2.2 ([13]) *If G_0 is a 3-connected cubic planar graph, then $\mathtt{cr}(G_0 + xy) = d^*_0(x, y)$.*

Riskin asked in [13] if Theorem 2.2 extends to arbitrary 3-connected planar graphs. In [12] it is shown that this is not the case: for every integer k, there exists a 5-connected planar graph G_0 and two vertices $x, y \in V(G_0)$ such that $\mathtt{cr}(G_0 + xy) \leq 11$ and $d^*_0(x, y) \geq k$. See also Gutwenger, Mutzel, and Weiskircher [6] for an alternative construction.

However, several extensions of Theorem 2.2 are possible. In particular, it is shown in [4] how to deal with graphs that are not 3-connected, and what happens when we allow vertices of arbitrary degrees.

Theorem 2.3 (Cabello and Mohar [4]) *If G_0 is a weighted planar graph and $x, y \in V(G_0)$, then*

$$d'_0(x, y) \leq \texttt{cr}(G_0 + xy) \leq d^*_0(x, y).$$

If G_0 is a cubic graph, then for every planar embedding of G_0, $d'(x, y) = d^*(x, y)$. Therefore, $d'_0(x, y) = d^*_0(x, y)$, and Theorem 2.3 implies Theorem 2.2.

Theorem 2.3 is also the main ingredient to improve the approximation factor in the algorithm of Hliněný and Salazar [7]; see Corollary 2.6.

A key idea in the proof is to show that $d^*_0(x, y)$ (respectively $d'_0(x, y)$) is closely related to the maximum number of edge-disjoint (respectively vertex-disjoint) cycles that separate x and y. The notion of the separation has to be understood in a certain strong sense. This result yields a dual expression for d^*_0 (respectively d'_0) and is used to show that $d^*_0(x, y)$ is closely related to the crossing number of $G_0 + xy$, while $d'_0(x, y)$ is in the same way related to the minor crossing number, $\texttt{mcr}(G_0 + xy)$, a version of the crossing number that works well with minors; see Bokal et al. [3].

As a complete surprise, Cabello and Mohar [4] proved that computing the crossing number of weighted near-planar graphs is NP-hard. Their reduction uses weights that are not polynomially bounded, and therefore it does not imply NP-hardness for unweighted graphs.

Despite examples and despite NP-hardness result for the weighted case, the following question may still have a positive answer:

Problem 2.4 ([4]) *Is there a polynomial time algorithm which would determine the crossing number of $G_0 + xy$ if G_0 is an unweighted 3-connected planar graph?*

As a corollary Cabello and Mohar [4] get a generalization of Riskin's Theorem 2.2.

Corollary 2.5 *If the graph $G_0 - x - y$ has maximum degree 3, then*

$$\texttt{cr}(G_0 + xy) = d'_0(x, y) = d^*_0(x, y).$$

In particular, the crossing number of $G_0 + xy$ is computable in linear time.

Another corollary is an approximation formula for the crossing number of near-planar graphs if the maximum degree is bounded.

Corollary 2.6 *If the graph $G_0 - x - y$ has maximum degree Δ, then $d'_0(x, y) \leq \texttt{cr}(G_0 + xy) \leq \frac{\Delta}{2} d'_0(x, y)$.*

Corollary 2.6 is an improvement of a theorem of Hliněný and Salazar [7] who proved analogous result with the factor Δ instead of $\Delta/2$.

A graph G is said to be *d-apex* if G has a vertex v of degree at most d such that $G - v$ is planar. Let us observe that every near-planar graph is essentially 2-apex (subdivide the "non-planar" edge).

Problem 2.7 ([4]) *Is there a result similar to Corollary 2.5 for 3-apex cubic graphs?*

3 Crossing sequences

Planarity is ubiquitous in the world of structural graph theory, and perhaps the two most obvious generalizations of this concept—crossing number, and embeddings in more complicated surfaces—are topics which have been thoroughly researched. Despite this, relatively little work has been done on the common generalization of these two: crossing numbers of graphs drawn on surfaces. This subject seems to have been introduced in [16], and studied further in [1]. Following these authors, we define for every nonnegative integer i and every graph G, the i^{th} *crossing number*, $cr_i(G)$, (and also the i^{th} *nonorientable crossing number*, $\tilde{cr}_i(G)$) to be the minimum number of crossings in a drawing of G on the orientable (nonorientable, respectively) surface of genus i. Observe that $cr_i(G) = 0$ (respectively, $\tilde{cr}_i(G) = 0$) if and only if i is greater or equal to the genus (resp., nonorientable genus) of G. This gives, for every graph G, two finite sequences of integers, $(cr_0(G), cr_1(G), \ldots, 0)$ and $(\tilde{cr}_0(G), \tilde{cr}_1(G), \ldots, 0)$, both of which terminate with a single zero. The first of these is the *orientable crossing sequence* of G, the second the *nonorientable crossing sequence* of G.

A natural question is to characterize crossing sequences of graphs. This is the focus of both [16] and [1]. If we are given a drawing of a graph in a surface S with at least one crossing, then modifying our surface in the neighborhood of this crossing by either adding a crosscap or a handle gives rise to a drawing of G in a higher genus surface with one crossing less. It follows from this that every orientable and nonorientable crossing sequence is strictly decreasing until it hits 0. This necessary condition was conjectured to be sufficient in [1].

Conjecture 3.1 (Archdeacon, Bonnington, and Širáň)
If $(a_1, a_2, \ldots, 0)$ *is a sequence of nonnegative integers which strictly decreases until* 0*, then there is a graph whose crossing sequence (nonorientable crossing sequence) is* $(a_1, a_2, \ldots, 0)$.

To date, there has been very little progress on this appealing conjecture. For the special case of sequences of the form $(a, b, 0)$, Archdeacon, Bonnington, and Širáň [1] constructed some interesting examples for both the orientable and nonorientable cases.

Theorem 3.2 (Archdeacon, Bonnington, and Širáň) *If* a *and* b *are integers with* $a > b > 0$*, then there exists a graph* G *with nonorientable crossing sequence* $(a, b, 0)$.

It has been believed by some that such a result cannot hold for the orientable case. For the most extreme special case $(N, N-1, 0)$, where N is a large integer, Salazar asked [14] if this sequence could really be the crossing sequence of a graph. The following quote of Dan Archdeacon illustrates why such crossing sequences are counterintuitive:

> If G has crossing sequence $(N, N-1, 0)$, then adding one handle enables us to get rid of no more than a single crossing, but by adding the second handle, we get rid of many. So, why would we not rather add the second handle first?

Recently, DeVos, Mohar and Šamal [5] proved a theorem that is an analogue of Theorem 3.2 for the orientable case, and its special case $a = N$, $b = N-1$ resolves Salazar's question [14].

Theorem 3.3 (DeVos, Mohar and Šamal [5]) *If a and b are integers with $a > b > 0$, then there exists a graph G whose orientable crossing sequence is $(a, b, 0)$.*

Quite little is known about constructions of graphs for more general crossing sequences. Next we shall discuss the only such construction we know of. Consider a sequence $\mathbf{a} = (a_0, a_1, \ldots, a_g)$ and define the sequence $(d_1, \ldots, d_g)$ by the rule $d_i = a_{i-1} - a_i$. If $\mathbf{a}$ is the crossing sequence of a graph, then, roughly speaking, d_i is the number of crossings which can be saved by adding the i^{th} handle. It seems intuitively clear that sequences for which $d_1 \geq d_2 \geq \cdots \geq d_g$ should be crossing sequences, since here we receive diminishing returns for each extra handle we use. Indeed, Širáň [16] constructed a graph with crossing sequence $\mathbf{a}$ whenever $d_1 \geq d_2 \geq \cdots \geq d_g$.

Constructing graphs for sequences which violate the above condition is rather more difficult. For instance, it was previously open whether there exist graphs with crossing sequence $(a, b, 0)$ where a/b is arbitrarily close to 1. The most extreme examples are due to Archdeacon, Bonnington and Širáň [1] and have a/b approximately equal to $6/5$. Although Theorem 3.3 gives a graph with every possible crossing sequence of the form $(a, b, 0)$, we don't know what happens for longer sequences. In particular, it would be nice to resolve the following problem which asks for graphs where the first s handles save only an epsilon fraction of what is saved by the $s+1^{st}$ handle.

Problem 3.4 ([5]) *For every positive integer s and every $\varepsilon > 0$, construct a graph G for which $cr_0(G) - cr_s(G) \leq \varepsilon\,(cr_s(G) - cr_{s+1}(G))$.*

For graph embeddings, the genus of a disconnected graph is the sum of the genera of its connected components. For drawing, this situation is presently unclear. If we have a graph which is a disjoint union of G_1 and G_2, then we can always "use part of the surface for G_1 and the other part for G_2", leading to

$$cr_i(G_1 \cup G_2) \leq \min_j \bigl(cr_j(G_1) + cr_{i-j}(G_2)\bigr)\,.$$

To the best of our knowledge, this inequality might always be an equality. More generally DeVos, Mohar and Šamal posed the following problem.

Problem 3.5 ([5]) *Let G be a disjoint union of the graphs G_1 and G_2, and let $\mathcal{S}$ be a (possibly nonorientable) surface. Is there an optimal drawing of G on $\mathcal{S}$, such that no edge of G_1 crosses an edge of G_2?*

This problem is trivially true when $\mathcal{S}$ is the plane. It is shown in [5] that it also holds when $\mathcal{S}$ is the projective plane.

Very recently, progress towards a solution of Problem 3.5 has been annonced. Two groups made progress in the case of the Klein bottle, while the authors of this note report some advance in the case of the torus.

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