Tenacity and Spectrum of a Graph

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The tenacity of a graph G, $T(G)$, is defined by $T(G) = \min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of $V(G)$, $\omega(G-S)$ be the number of components of $G-S$ and $\tau(G-S)$ be the number of vertices in the largest component of the graph induced by $G-S$.

The largest number of vertices in any such set is called the vertex independence number of G and is denoted by $\alpha(G)$.

k-tree of a connected graph G is a spanning tree with maximum degree at most k. In this paper we show that if $T(G) \geq \frac{\tau(G-S)}{\omega(G-S)} + \frac{1}{k-2}$, for any subset S of $V(G)$, with $k \geq 3$, then G has a k-tree. Also we choose the spectrum of a graph, G_n^m, which is n-connected and we calculate the tenacity of G_n^m.

Theorem 1. *If*

$$T(G) \geq \frac{\tau(G-S)}{\omega(G-S)} + \frac{1}{k-2}, \qquad \text{with } k \geq 3, \qquad (1)$$

for any vertex cutset S of G, then G has a k-tree.

The Tenacity of G_n^m. We may assume G_n^m is labeled by $0, 1, 2, \cdots, m$. Let n be even, $n = 2r$ and two vertices i, j are adjacent, if $i - r \leq j \leq r$ (where addition is taken modulo m). Now let n and m be odd, $(n > 1)$. Let $n = 2r + 1$, $(r > 0)$. Then G_{2r+1}^m is constructed by first drawing G_{2r}^m, and adding edges joining vertex i to veretex $i + \frac{m+1}{2}$ for $1 \leq i \leq \frac{m-1}{2}$. Note that vertex 0 is adjacent to both vertices $\frac{m-1}{2}$ and $\frac{m+1}{2}$.

Theorem 2. *Graph G_n^m is n-connected.*

Theorem 3. $T(G_{2r}^m) = r + \frac{1+\lceil \frac{s}{k} \rceil}{k}$.

Through the rest of this paper we will let $n = 2r$ or $n = 2r+1$ and $m = k(r+1)+s$ for $0 \leq s \leq r+1$. So we can indicate that $m \equiv s \ mod(r+1)$ and $k = \lfloor \frac{m}{r+1} \rfloor$. Also we assume that the graph G_n^m is not complete, $n + 1 < m$.

Theorem 4. *Let G_n^m be the graph with m and n odd, $n = 2r + 1$, then*

$$r + \frac{1 + \lceil \frac{s}{k} \rceil}{k} \leq T(G_n^m) \leq \begin{cases} r + \frac{s+1}{k} & if m \not\equiv 1 \ mod(n+1) \\ \frac{kr+s+2}{k-1} & if \ m \equiv 1 \ mod(n+1) \end{cases}$$

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LEMMA 5. *Let G_n^m be the graph with m odd, $n = 2r+1$, $r \geq 2$, $1 < s < r+1$, $s < k$, and k is even. Then there is an cutset A with kr+1 elements such that number of components is $\omega(G_n^m - A) = k$, and the largest component is $\tau(G_n^m - A) = 2$.*

THEOREM 6. *Let G_n^m be the graph with $n = 2r+1$, m odd, k even, $k > 2$, $1 < s < r+1$ and $s < k$. Then*

$$r + \tfrac{2}{k} \leq T(G_n^m) \leq r + \tfrac{3}{k}.$$

LEMMA 7. *Let G_n^m be the graph with $n = 2r+1$, m odd, k even, $k > 2$, $1 < s < r+1$, and $s > k$ where $s = ak + b$, for some a and b, $0 < b < k$, then*

$$r \geq \begin{cases} 5 & \text{for a odd} \\ 9 & \text{for a even.} \end{cases}$$

Note that if r is even then the bounds in the above lemma can be increased by 1.

LEMMA 8. *Let G_n^m be the graph with $n = 2r+1$, m odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$ where $s = ak + b$, for some a and b, $0 < b < k$, $a + 1 < \frac{r}{2}$.*

Finally, we have the following theorem:

THEOREM 9. *Let G_n^m be the graph with $n = 2r+1$, m odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$. Write $s = ak + b$ for some a and b and $k = 2q$ for some q. Then*

$$r + \frac{a+2}{k} \leq T(G_n^m) \leq \begin{cases} r + \frac{a+2+z}{k}, & \textit{where} \quad z = \frac{a+\frac{b}{2}-1}{q-1} \\ r + \frac{a+3+z}{k} & \textit{where} \quad z = \lfloor \frac{a+\frac{b}{2}-1}{q-1} \rfloor. \end{cases}$$

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