The 2-extendability of graphs on the projective plane, the torus and the Klein bottle

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A pair or a set of edges is said to be *independent* or is called a *matching* if no two edges belonging to it have a common end. In particular, an independent set of edges which covers all vertex in a graph is called a *perfect matching.* A graph G with $|V(G)| \geq 2k+2$ is said to be k*-extendable* if any independent set of k edges in G is contained in a perfect matching.

Plummer [1] has introduced this notion of k-extendability of graphs and has proved the following theorem:

Theorem 1. (Plummer [1]) *Every* 5*-connected planar graph of even order is* 2*-extendable.*

Futhermore, he has characterized those 4-connected maximal planar graphs that are 2-extendable, specifying the structure called "generalized butterflies".

Corresponding to these theorems, we shall prove the following theorems on the 2-extendability of graphs on surfaces concerning the connectivity:

Theorem 2. *Every* 5*-connected graph of even order embedded on the projective plane is* 2*-extendable.*

Theorem 3. *Every* 6*-connected graph of even order embedded on the torus is* 2*-extendable.*

Theorem 4. *Every* 6*-connected graph of even order embedded on the Klein bottle is* 2*-extendable.*

The existence of a generalized butterfly actually prevents a graph from being 2-extendable, but any 5-connected graph cannot contain it. Thus, there will be another forbidden structure for 5-connected graphs to be 2-extendable. Carrying out topological arguments on the torus, we can prove the following theorem in particular:

Theorem 5. *A* 5*-connected graph of even order embedded on the torus is* 2*-extendable if and only if it has none of the structures depicted in Figure 1*

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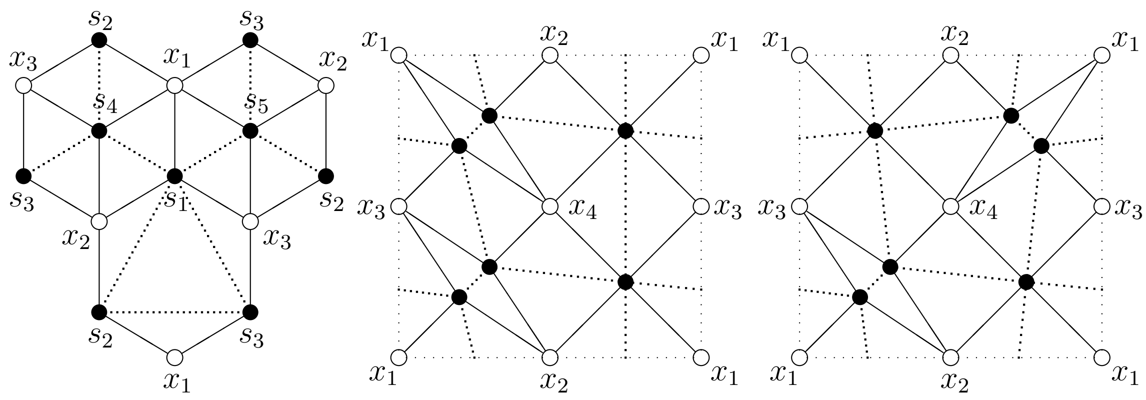

(I) (II) (II)

Figure 1 Forbidden structures for the 2-extendability of 5-connected graphs on the torus

To obtain an actual graph which are not 2-extendable, replace each of white vertices with a connected planar graph of odd order and choose additional edges from edges drwan by dotted lines so that they include at least one independent pair of edges. The edges between a white vertex and a black vertex may split into several edges with a common back end. Furthermore, the resulting graph should be simple and 5-connected. For example, all dotted edges in (II) cannot be chosen together since the vertical dotted line segment in the right half forms multiple edges between two black vertices. Therefore, if one wants a triangulation on the torus, then only (I) and (III) are available.

References

[1] M.D. Plummer, Extending matchings in planar graphs IV, *Discrete Math.* **109** (1992), 207–219.