Reversing a polyhedral surface by origami-deformation

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A *flexatube* is a puzzle invented by Arthur H. Stone in 1939. It is made from a cardboard cubical box by removing a pair of opposite faces, and then by making creases on each square face along the pair of diagonals. A flexatube can be turned inside out by a series of folds along the edges of the squares and the creased diagonals. To turn a flexatube inside out is an interesting and tantalizing puzzle.

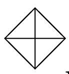
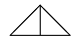
Stimulated by the flexatube, I sought other interesting variations of flexatube, and found one in 2006, which I called a *rhombotube*. It is obtained from a hollow rhombohedron whose six faces are rhombi with interior angles 60° and 120° by removing a pair of opposite faces, and then, by making creases along the pairs of diagonals of the remaining four faces. It can be also turned inside out by a series of folds along the creases and edges. To reverse a rhombohedron is a challenging puzzle. A solution I found is exquisite and complicated.

Motivated by these reversible tubes, I started to consider the reversibility for general polyhedral surfaces. A polyhedral surface M is a 2-dimensional manifold in R^3 made from a finite number of cardboard polygons by attaching along their edges. The materials, cardboard polygons, are supposed to be very thin, and their thickness is regarded to be 0. Each polygon of M is called simply a *face* of M. A *subdivision* of M is a polyhedral surface obtained from M by subdividing faces of M into small polygons.

An origami-deformation of a polyhedral surface is a deformation that corresponds to a series of folds along the edges or creases:

Definition. An *origami-deformation* of a polyhedral surface $M \subset R^3$ is a continuous motion $f_t : M \to R^3$ $(0 \le t \le 1)$ of M such that (1) f_0 is the inclusion map, (2) for each face of M, the induced motion of the face is a rigid motion, and (3) two faces may touch or overlap during the motion, but they never go through each other.

Note that since two faces may overlap, $f_t : M \to R^3$ is not always an embedding for every $t \in [0, 1]$. However, the condition (3) implies that if $f_1 : M \to R^3$ is an embedding, then M can be changed to $f_1(M)$ by a deformation through 'topological embeddings', that is, the inclusion map f_0 and the embedding f_1 are isotopic.

To see effects of origami-deformations, consider, for example, the polyhedral surface  obtained by subdividing a unit square with the pair of diagonals. It can be turned by origami-deformation into a 2-sheet-triangle  (by folding along the horizontal diagonal). Then, since the thickness of the resulting 2-sheet-triangle

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is zero, it can be turned, by folding the two sheets together along the (superposed) median, into a 4-sheet-triangle ⊿.

The reversibility for a general polyhedral surface is defined as follows.

Definition. A polyhedral surface M is called *reversible* if there is an origami-deformation $f_t : M \to R^3$ $(0 \le t \le 1)$ such that $f_1(M)$ is a mirror image of M with respect to a plane, and the correspondence $M \ni x \mapsto f_1(x) \in f_1(M)$ is the reflection map. If a subdivision M' of M is reversible, then M is called *subdivision-reversible.*

Concerning the reversibility of general polyhedral surfaces, the following holds. (Note that under the above definition, one may consider the reversibility even for a non-orientable surface.)

Theorem 1. *If a polyhedral surface contains a 2-component-link (α, β) with non-zero linking number, then the surface is not subdivision-reversible.*

It is known (Conway and Gordon 1983, Sachs 1983) that every embedding of the complete graph K_6 in R^3 contains a 2-component link with nonzero linking number. Since K_6 is embeddable in the Möbius band, and also embeddable in any orientable polyhedral surface of positive genus, the next corollary follows.

Corollary 2. *Every subdivision-reversible polyhedral surface is homeomorphic to a subset of a 2-dimensional sphere.*

Theorem 3. *If a polyhedral surface has an interior vertex at which the sum of face angles is less than 2π, then the surface is not subdivision-reversible.*

Thus, an open box (that is, a rectangular tube closed on the bottom) is not subdivision-reversible because the sum of face angles at each bottom vertex is equal to $3\pi/2$, which is less than 2π.

Let M be a polyhedral surface and F be a face of M. Cut out a rectangle from F and attach a rectangular tube at the hole along the boundary of the rectangular hole, perpendicularly to the face. Then, a new polyhedral surface is obtained. This new surface is called a surface obtained from M by a *tube-attachment operation.*

Theorem 4. *If M is subdivision-reversible, then every polyhedral surface obtained from M by applying tube-attachment operations one after another, is subdivision-reversible.*

Some examples and problems will be also given in the talk.