Generalized Polyhedral Suspensions

Abstract of proposed lecture at the 20th Workshop on Topological Graph Theory in Yokohama

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1. Introduction

We have discovered a new class of polyhedra — bipyramids of arbitrary genus. We describe our construction, present unfoldings of bipyramids of genus 1 and 2, with the rules of gluing.

In the lecture we will also present some of the current interesting directions in our research as well as some possible future directions. Our work on those is currently in progress and we here also preview some of the results based on our recent work. In particular, we have discovered a new *regular* polyhedron as a torus type bipyramid geometrically re-embedded from Euclidean space E^3 into E^4 (Section 6). Furthermore, in Section 4 we obtain as a byproduct a reinforcement of a classical theorem [R] of Ringeisen which is in fact a plain corollary of our construction (Theorem 4).

Considered in this lecture are polyhedra with triangular faces, embedded in Euclidean space. Recall that the term "embedding" excludes self-intersections — that is, the triangles realizing the faces must have disjoint interiors. (In the case where a polyhedron has nontriangular faces, diagonals can be added to triangulate them.) In what follows, by a *triangle* we will mean a triangular face of a polyhedron.

A n*-complex* will mean an n-dimensional simplicial complex. Associated with a polyhedron P is its *geometric 2-complex,* $K^2(P)$; it consists of the vertices, edges, and triangles of P. The corresponding topological 2-complex is referred to as a *topological 2-complex of* P. The term "2-complex of P" can carry both meanings, topological and geometric.

By S_g we will denote a closed, compact, orientable, 2-dimensional surface of genus g. When the underlying space of $K^2(P)$ is homeomorphic to S_g, we call P a *polyhedron of genus* g. In particular, polyhedra of genus 0 and 1 are *sphere type* and *torus type polyhedra,* respectively.

A *suspension* is a polyhedron in which all but two vertices lie in one plane, the *equatorial plane.* The *equatorial 2-complex of a suspension* is the 2-complex consisting of the equatorial vertices, edges, and triangles of the suspension. The two vertices not in the equatorial plane are located above and below that plane, respectively. They are designated as the north pole, N, and the south pole, S, respectively. Especially, in this lecture we are concerned with *bipyramids* — that is, polyhedral suspensions with the additional restriction that both N and S be adjacent to *each* vertex in the equatorial plane.

It is obvious how to generate sphere type bipyramids. They have a circuit of arbitrary length as the equatorial 2-complex (it degenerates to a 1-complex in this simplest case). The existence of bipyramids of higher genera, g, is not obvious at all. For $g = 1$, a torus type bipyramid, a construction can be found in [BL]. (Moreover, it is shown in [BL] that many triangulations of the torus with few vertices can be realized as polyhedral suspensions.) A number of geometers have been very interested in whether there exist polyhedral suspensions of topological types other than the sphere and the torus. We will answer this question affirmatively, establishing the existence of generalized polyhedral suspensions.

Our construction is presented in Section 3. We discuss the relationship between suspensions and 2-complex planarity in Section 2. We discuss properties of bipyramids in Section 4 and give adequate instructions for assembling their models in Section 5. We give concluding remarks on our plans for current and prospective research in Section 6.

2. Planarity of 2-complexes

The topic which we introduce in this section is closely related to the structure of suspensions as we will see shortly.

A topological n-complex K^n, $n \in \{1,2\}$, is said to be *topologically planar* if it can be embedded in the plane. Recall that the term "embedded" means that the simplexes of K^n are represented by *interiorly disjoint* curves (1-siplexes) and triangular regions (2-simplexes) in the plane. K^n is said to be *geometrically planar* if it is topologically planar and can be realized by a geometric 2-complex in the Euclidean plane E^2 – that is, with all its 1-simplexes represented by straight line segments. It should be stressed that we admit unbounded triangles (the outer region), so any 2-complex whose underlying space is homeomorphic to the sphere is topologically *and geometrically* planar.

Topologically planar 1-complexes have been characterized by Kuratowski's theorem [K]. Furthermore, Fáry [F] proved that each topologically planar 1-complex is geometrically planar. We now state and prove a 2-dimensional analog of Fáry's theorem.

Theorem 1. *A 2-complex K^2 is topologically planar if and only if K^2 is geometrically planar.*

Proof: In fact, we need to prove that the topological planarity of K^2 implies its geometric planarity. We give here the proof for the case in which the graph $G(K^2)$ of K^2 is 3-connected. By Fáry's theorem [F], we can embed the graph $G(K^2)$ in the plane with all of its edges represented by straight line segments. On the other hand, by Whitney's theorem [W], that graph embedding is combinatorially unique up to what region is chosen as the outer one. Hence, thanks to the topological planarity of K^2, adding the 2-simplexes to that graph embedding won't cause any impediments for 2-dimensional planarity. ■

Observe that a topological 2-complex K^2 is the 2-complex of some suspension if and only if:

(i) the underlying space of K^2 is homeomorphic to an orientable, closed surface,

and

(ii) there exists a planar subcomplex of K^2 that contains all but two vertices of K^2.

It is not generally true that a 2-complex is planar whenever its graph is planar. As a counterexample, take a sphere type bipyramid whose equatorial graph is a circuit of length 3 and fill that circuit with a 2-simplex. The so-obtained 2-complex is non-planar. Note that its graph is planar and has connectivity equal to 3.

Theorem 2. *A 2-complex is planar if its graph is 4-connected and planar.*

Proof: Firstly embed the graph of the 2-complex in the plane with all of its edges represented by straight line segments. This is possible by Fáry's theorem [F]. Now add the 2-simplexes of the 2-complex. By the 4-connectivity hypothesis, there are no separating circuits of length 3, whence there are no impediments to embeddability, and the statement follows. ■

The *star* of v, denoted $\mathrm{st}(v)$, in K^2 is defined to be the minimum subcomplex of K^2 that contains each simplex incident with v.

We now conjecture two 2-complex planarity criteria.

Conjecture 1. *A 2-complex* K^2 *is planar if and only if its graph* $G(K^2)$ *is planar and the star of each vertex in* K^2 *is planar.*

Conjecture 2. *A 2-complex* K^2 *is planar if and only if the graph of the 2-complex obtained from* K^2 *by stellar subdivision of each 2-simplex is planar.*

It would be also very interesting to characterize planar 2-complexes in terms of forbidden 2-complexes. This would be a 2-dimensional analog of Kuratowski's theorem [K].

3. Construction

In this section we constructively establish the existence of generalized polyhedral suspensions — more specifically, bipyramids of arbitrary genus g. We will denote them by B_g.

Observe that the underlying space of a 2-complex K^2 is isomorphic to a closed surface if and only if the star of each vertex in K^2 represents a disk (2-dimensional). This condition ensures the absence of singular vertices and, moreover, that each edge is incident with precisely two triangles.

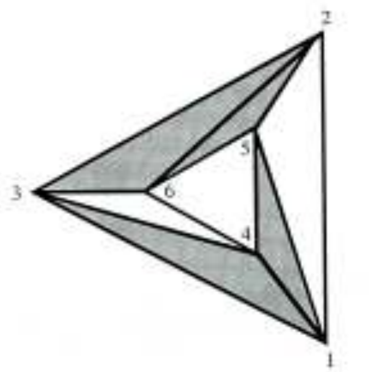

Figure 1. Basic equatorial template for B_1.

We now reproduce the construction [BL] for building a torus type bipyramid B_1 with eight vertices. The equatorial 2-complex is shown in Figure 1, where its 2-simplexes are shaded. The two vertices not on the equatorial plane are the north pole N and the south pole S. These vertices are placed above and below the equatorial plane, respectively. We add the triangles determined by N and the edges of the circuit 6, 4, 5, 2, 1, 3, 6 as well as the triangles determined by S and the edges of the circuit 6, 5, 1, 2, 3, 4, 6**.** Both circuits are Hamilton circuits of the equatorial graph. It can be readily verified that there are exactly two triangles meeting each edge and that the star of each vertex is homeomorphis to a disk. Therefore we have a closed, orientable surface. Using Euler's equation, one can check that that surface is a torus. Therefore the so-obtained polyhedron is a torus type bipyramid.

Now we state and prove our basic result.

Theorem 3. *There exists a bipyramid of genus* g *for each positive integer* g.

Proof: As mentioned above, bipyramids of genus 1 are already known [BL]. So our job is to construct a bipyramid of genus g for a given $g \geq 2$. As the equatorial 2-complex we take g copies of the template of Figure 1 and link them together in cyclic order as shown in Figures 2 and 3, respectively. Those figures present the equatorial 2-complexes for $g = 2$ and $g = 3$, respectively. This construction generalizes for an arbitrary g in a natural fashion.

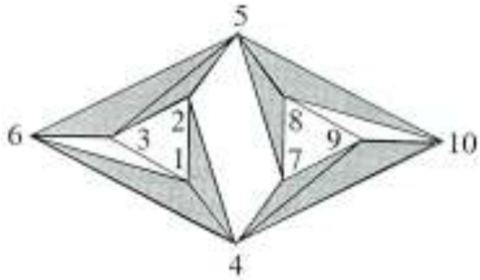

Figure 2. Equatorial 2-complex for a bipyramid B_2 of genus 2.

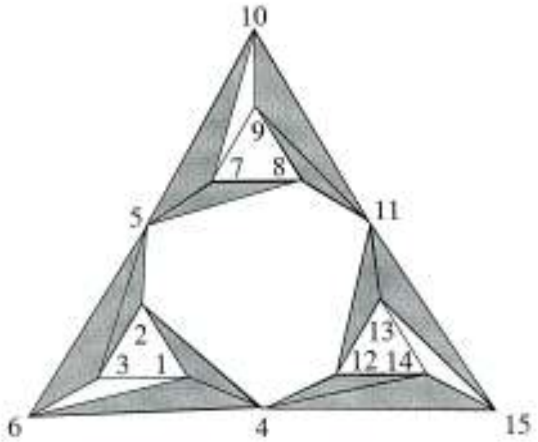

Figure 3. Equatorial 2-complex for a bipyramid B_3 of genus 3.

Now we proceed to add nonequatorial triangles to build a desired bipyramid. They are determined by certain equatorial edges and the vertex N or S. Our construction is general in nature, but, for convenience, we will explain it specifically for B_2. For this, consider the equatorial circuits Z_1 = 6, 5, 7, 9, 8, 10, 4, 2, 3, 1, 6 and Z_2 = 6, 4, 7, 8, 9, 10, 5, 2, 1, 3, 6 in the 2-complex of Figure 2. Note that these circuits share two edges, $\{3,1\}$ and $\{8,9\}$. (Those circuits also appear in Figure 7.) To complete construction of the bipyramid B_2, as nonequatorial triangles we choose the triangles determined by the pole N and the edges of Z_1 as well as the triangles determined by the pole S and the edges of Z_2. Observe that, for an arbitrary g, Z_1 and Z_2 satisfy each of the following three conditions.

Condition 1: Each equatorial edge either:

(a) appears as a side of exactly two adjacent equatorial triangles,
or
(b) appears as a side of exactly one equatorial triangle and in either Z_1 or Z_2, but not both,
or
(c) does not appear as a side of any equatorial triangle but appears in both Z_1 and Z_2.

Condition 2: Both Z_1 and Z_2 are Hamilton circuits of the equatorial graph — that is, both span all vertices of the equatorial 2-complex.

Condition 3: At each equatorial vertex v, the circuits Z_1 and Z_2 are not placed in either way as shown in Figure 4.

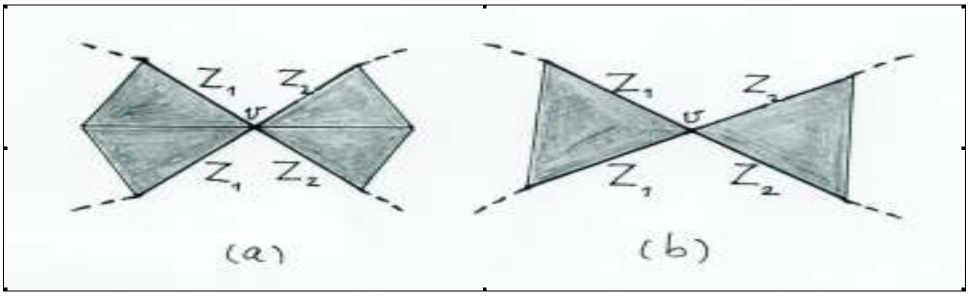

Figure 4. Forbidden situations where singular vertices occur.

Such a pair of circuits can be obviously found in the equatorial 2-complex for an arbitrary g.

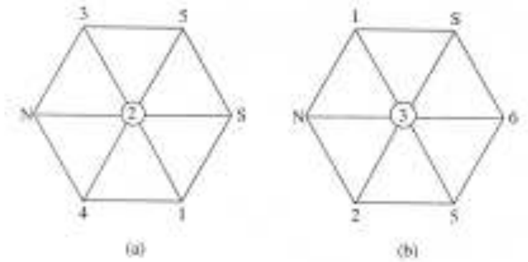

Figure 5. Stars of vertices 2 and 3.

Conditions 1 and 2 ensure that each edge appears as a side of precisely two triangles of the polyhedron constructed. Furthermore, Condition 3 ensures the absence of singular vertices in the equatorial plane. For example, consider the equatorial vertices 2 and 3 in Figure 2. Figure 5(a) shows the triangles as they appear in cyclic order around the vertex 2. Those triangles form a disk, which means that $\mathrm{st}(2)$ is a disk and therefore the vertex 2 is a nonsingular vertex. Similarly, Figure 5(b) shows the triangles as they appear around the equatorial vertex 3. Again, the triangles form a disk, which means that the vertex 3 is nonsingular. Finally, the nonsingularity of the nonequatorial vertices, N and S, is guaranteed by Condition 2.

Therefore, what we have constructed is indeed a polyhedron representing a closed surface and, moreover, that polyhedron is a bipyramid. This specific polyhedron constructed is denoted by B_g. Finally, by construction, the polyhedron is indeed embedded in E^3 and therefore represents an orientable surface.

It remains to show that the genus of B_g is equal to g — the number of templates. To accomplish this, we shall now show that inserting one template in the equatorial 2-complex increments the genus by 1, for $g \geq 2$. It is a matter of routine counting to calculate the number of vertices V, edges E, and triangles F of B_g in terms of g:

$V = 5g + 2$, $E = 21g$, and $F = 14g$.

Then, by Euler's equation, $V - E + F = 2 - 2g$, the genus of B_g is indeed is equal to g. The proof is complete. ■

4. Properties of bipyramids

Let us begin with a topological question for surfaces: What is the minimum number of disks needed to cut out from S_g so that the resulting surface could stretch in the plane without breaking or overlapping? For a simple topological reason, removing only one disk is not enough, for $g \geq 1$. Therefore the answer to the question is 2 because the removal of the stars of the two poles from B_g leads to a planar (equatorial) 2-complex.

Interestingly, the genus of the graph of B_g is equal to g, while its thickness is equal to 2 for any g. It would be interesting to find the chromatic and crossing numbers of the graphs of B_g as well as other important invariants.

It is also interesting to observe the properties of the equatorial graphs, denoted $G_{eq}(B_g)$, of B_g. More specifically, these graphs are planar for any g, but the graph $G_{eq}(B_g)$ 2-cell embeds in S_g, whence $0 = \gamma(G_{eq}(B_g)) \leq g \leq \gamma_M(G_{eq}(B_g))$. Hence, by a theorem of Nordhaus-Stewart-White [NSW], the graph $G_{eq}(B_g)$ is 2-cell embeddable in each surface in a row from S_0 through to S_g. As a graph-theoretical byproduct, we therefore re-discover the existence of planar graphs of arbitrarily large maximum genus. This is a classical theorem [R] of Ringeisen. We can even strengthen Ringeisen's theorem. For that, define the *polygonal maximum genus of a graph* G to be the maximum h such that G can 2-cell embed on S_h without repeated vertices on the boundary of each 2-cell. Then our result is as follows.

Theorem 4. *There exist planar graphs of arbitrarily large polygonal maximum genus.*

For more properties, observe that the graph of each B_g is a Hamilton graph and that each B_g corresponds to an irreducible triangulation of S_g in the sense of Negami-Lawrencenko [LN]. Also observe that, in the case in which the poles of B_g are placed symmetrically with respect to the equatorial plane, the reflection with respect to that plane gives another suspension which is the reverse of the original B_g and is congruent to B_g. In other words, such a reflection takes the underlying surface S_g inside out. It may be interesting to look into this phenomenon along the lines of Maehara's approach [M].

5. Physical models and computer images

In this section we show unfoldings of the bipyramids B_1 and B_2 constructed in the preceding section and describe how to build their models. Their cardboard models will be demonstrated in the lecture as well as computer-generated images.

Table 1: Equatorial measurements for $g = 1$

Edge	Length	Edge	Length
45	18.6 cm	63	7.1 cm
56	18.6 cm	52	7.1 cm
64	18.6 cm	32	6.3 cm
61	12.9 cm	21	6.3 cm
24	12.9 cm	13	6.3 cm

Table 2: Template measurements for $g = 1$

Template	Side 1	Side 2	Side 3
A	18.6 cm	18.6 cm	18.6 cm
B	6.3 cm	17.1 cm	17.1 cm
C	7.1 cm	18.6 cm	17.1 cm
D	12.9 cm	18.6 cm	17.1 cm

Firstly, we give instructions for constructing the torus type bipyramid B_1. Draw the equatorial 2-complex depicted in Figure 1 on cardboard, with measurements as indicated in Table 1. Cut out the shape bounded by the circuit 1, 3, 2, 5, 1. Then remove and discard the fragment bounded by the circuit 3, 4, 5, 6, 3. What remains is the equatorial 2-complex for B_1 with the edges $\{1,2\}$ and $\{4,6\}$ removed.

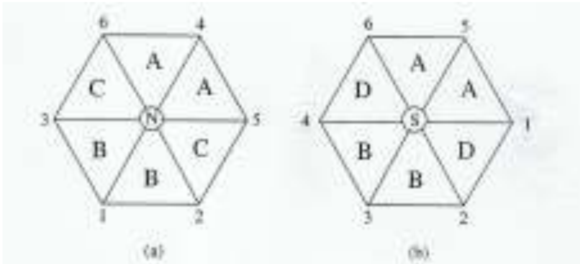

Figure 6. Placement of templates around N and S for g = 1.

Secondly, we paste the triangles connecting the equatorial 2-complex to the poles N and S. These triangles are taken from the set of four templates, named A, B, C, and D, with measurements as indicated in Table 2. Figure 6 shows charts (a) and (b) which list the templates in cyclic order they occur around N and S, respectively. The columns indicate the approximate side lengths in centimeters.

Finally, paste the triangles listed in Table 2 and Figure 6 to the equatorial 2-complex using scotch tape. Also tape together the triangles along their common sides. With all of the triangles pasted, we obtain a complete model of B_1.

Table 3: Equatorial measurements for $g = 2$

Edge	Length	Edge	Length
6 5	18.6 cm	6 3	7.1 cm
5 10	18.6 cm	4 1	7.1 cm
10 4	18.6 cm	4 7	7.1 cm
4 6	18.6 cm	5 8	7.1 cm
6 1	12.9 cm	9 10	7.1 cm
2 4	12.9 cm	1 2	6.3 cm
5 3	12.9 cm	2 3	6.3 cm
5 7	12.9 cm	3 1	6.3 cm
4 9	12.9 cm	7 8	6.3 cm
8 10	12.9 cm	8 9	6.3 cm
5 2	7.1 cm	9 7	6.3 cm

Table 4: Template measurements for $g = 2$

Template	Side 1	Side 2	Side 3
E	18.6 cm	24.6 cm	20.8 cm
F	7.1 cm	24.6 cm	20.7 cm
G	12.9 cm	19.0 cm	20.8 cm
H	7.1 cm	19.0 cm	20.8 cm
I	6.3 cm	19.0 cm	19.0 cm
J	6.3 cm	19.0 cm	20.7 cm
K	12.9 cm	19.0 cm	24.6 cm

The process of building B_2 is similar to the process for the torus type.

On one cardboard sheet**,** draw the equatorial 2-complex depicted in Figure 2, with measurements as indicated in Table 3. Then cut out the shape bounded by the circuit 6, 5, 10, 4, 6. Remove and discard the shapes bounded by the circuit 1, 2, 3, 6, 1, the circuit 7, 8, 10, 9, 7, and the circuit 5, 7, 4, 2, 5. This gives the equatorial 2-complex of B_2 with the edges $\{1,3\}$ and $\{8,9\}$ removed.

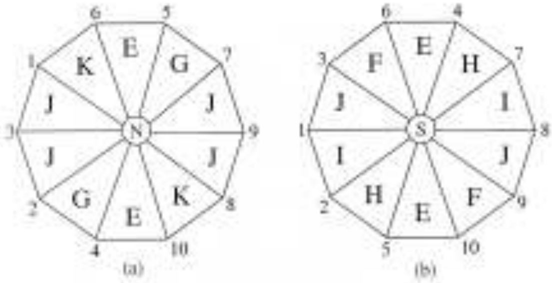

Figure 7. Placement of templates around N and S for g = 2.

The technique for attaching the triangles connecting the equatorial 2-complex to the poles is essentially the same as for the torus type. We now use seven triangular templates, named E, F, G, H, I, J, and K. Table 4 gives their measurements. Figure 7 indicated the rules for placement of the templates.

In addition to the cardboard models, we are going to demonstrate their computer-generated images on our website (currently under construction) devoted to polyhedral suspensions. In particular, it will present computer-generated images of B_1 and B_2, respectively. We have designed a computer program that generates bipyramidal models for a given $g \geq 2$. The program is written in the Scheme programming language, a descendant of the LISP language. It outputs a VRML file. (VRML is a computer language used to create 3D models.)

6. Summary of current and prospective research

In addition to Sections 2 and 4, we here very briefly address three more topics for research.

4.1. Regular polyhedra. Modifying our construction of B_1 in Section 3, we have geometrically realized a triangulated torus with 8 vertices as a *regular* polyhedron in E^4.

Theorem 5. *There exists a regular toroidal polyhedron in* E^4.

4.2. Rigidity and volumes of suspensions. Can we say anything about rigidity for suspensions of higher genera? Connelly proved [C] that the generalized volume of a non-rigid suspension is necessarily equal to zero. (To find the volume of B_g is an amusing puzzle on elementary geometry.)

4.3. Enumeration of bipyramids. One can obtain more bipyramids by finding more pairs of Hamilton circuits satisfying Conditions 1, 2, and 3 (Section 3), in the equatorial 2-complex of B_g (Figures 1, 2, 3). It would be interesting to calculate the precise number of such pairs. Alternatively, we can calculate the number of bipyramids with the same graph by using an enumerative formula [CKL] of Kwak-Lawrencenko-Chen. Then it would be interesting to compare results against each other.

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