Algorithms for Finding an Induced Cycle in Planar Graphs and Bounded Genus Graphs

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In this talk, we consider the problem for finding an induced cycle passing through k given vertices, which we call the *induced cycle problem* (ICP). The significance of finding induced cycles stems from the fact that precise characterization of perfect graphs would require structures of graphs without an odd induced cycle of length more than three (called an odd hole), and its complement. There has been huge progress in the recent years, especially, the Strong Perfect Graph Conjecture was solved in [3]. Concerning recognition of perfect graphs, there had been a long-standing open problem for detecting an odd hole and its complement, and finally this was solved in [2].

Unfortunately, the problem of finding an induced cycle passing through two given vertices is NP-complete in a general graph [1]. However, if the input graph is constrained to be planar and k is fixed, then the induced cycle problem can be solved in polynomial time [4, 5, 6]. In particular, for the case $k = 2$, McDiarmid, Reed, Schrijver and Shepherd [7] gave the following result.

Theorem 1. (McDiarmid et al. [7]) *Suppose an input graph is planar. Then there is an* $\mathrm{O}(n^2)$ *time algorithm for the ICP with* $k = 2$*, where* n *is the number of vertices of the given graph.*

Our main results are to improve their result in the following sense.

1. The number of vertices k is allowed to be non-trivially super constant number, up to $k = \mathrm{o}((\frac{\log n}{\log \log n})^{\frac{2}{3}})$.
2. The time complexity is linear if the given graph is planar and k is fixed.
3. The above results are extended to graphs embedded in a fixed surface.

More precisely, our first two results are the followings:

Theorem 2. *If* $k = \mathrm{o}((\frac{\log n}{\log \log n})^{\frac{2}{3}})$*, then the ICP in planar graphs can be solved in* $\mathrm{O}(n^{2+\varepsilon})$ *time for any* $\varepsilon > 0$*, where* n *is the number of vertices of the input graph.*

Theorem 3. *The ICP is solvable in linear time when* k *is fixed and the input graph is planar.*

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We note that the linear time algorithm (the second result) is independent from the first result. Let us observe that if k is as a part of the input, then the problem is NP-complete, and so we need to impose some condition on k.

We also show that the above results can be extended to graphs embedded in a fixed surface. These results generalize Theorems 2 and 3.

Theorem 4. *Let G be a graph with n vertices embedded in a fixed surface. If $k = \mathrm{o}((\frac{\log n}{\log\log n})^{\frac{2}{3}})$, then the ICP in G can be solved in $\mathrm{O}(n^{2+\varepsilon})$ time for any $\varepsilon > 0$. If k is fixed, the ICP in G is solvable in linear time.*

Let us point out that our algorithm is the first polynomial time algorithm for the ICP for the bounded genus case.

Our proofs of the above results basically follow the same line of the proof of the disjoint paths problem by Robertson and Seymour [10], together with some arguments in [8, 9], which improves the time complexity of the algorithm of the disjoint paths by Robertson and Seymour ($\mathrm{O}(n^3)$ time algorithm) to linear time when an input graph is planar. However, our cycle must be induced, so some of arguments must be extended to induced paths, which needs much more involved arguments. In addition, we are also interested in the case when k is as a part of the input. Therefore, we need to sharpen the function of k. This needs nontrivial amount of work, since the proofs in [9, 10] do not care much about sharpening the hidden constant of k. Price to pay is to need to analyze the structure of planar graphs and graphs embedded in a fixed surface more closely.

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