Knots contained spatial embeddings of complete graphs

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Let K_n be a complete graph with n vertices. Conway and Gordon showed that every spatial embedding of K_7 contains a non-trivial knot [1]. However, their theorem does not tell us the sharp relation between the number of vertices of a given complete graph and the crossing number of a knot contained in a spatial embedding of the graph. Here, we will pay attention to the crossing number of knots which are contained by spatial embeddings of complete graphs with vertices being greater than or equal to 8.

In a joint paper with Toshifumi Tanaka, we have constructed a spatial embedding of a graph as follows. The spatial embedding of a graph is a linear spatial embedding $\bar{f}: K_n \to \mathbb{R}^3$ such that each vertex of K_n is mapped into the moment curve $H := \{(\cos\theta, \sin\theta, 2\pi - \theta) \in \mathbb{R}^3; 0 \leq \theta \leq 2\pi\}$ such that i-th vertex of K_n is mapped on the coordinate $\left(\cos\dfrac{2i\pi}{n}, \sin\dfrac{2i\pi}{n}, 2\pi - \dfrac{2i\pi}{n}\right)$ for $i = 1, \cdots n$. Let $T(p,q)$ be the torus knot of type (p,q) [2].

Theorem 1. (K-Tanaka [3]) *Let n be an integer with $n \geq 4$. The spatial graphs $\bar{f}(K_{2n-1})$ and $\bar{f}(K_{2n})$ contain $T(2n-5, 2)$.*

Here, we conjecture that every spatial graph of K_9 will contain knots with the crossing number being greater than or equal to 5. If it is true, it is a direct generalization of this conjecture. Let $\tilde{f}(K_9)$ be a spatial embedding of K_9 which is that of $\bar{f}(K_9)$ after one crossing change.

Theorem 2. *The spatial graph $\tilde{f}(K_9)$ contains knots with the crossing number being greater than or equal to 5.*

Here, we give an outline of the proof. Let $v_0, v_1, \cdots, v_8$ be vertices of the spatial graph $\bar{f}(K_9)$. Here $z(v_i)$ is less than $z(v_{i+1})$, where $z(v_i)$ is the z-coordinate of v_i. Let $E(v_i, v_j)$ be the edge connecting v_i and v_j. The spatial graph $\bar{f}(K_9)$ contains knots with the crossing number being greater than or equal to 5 which is the cycle $v_0, v_2, v_4, v_6, v_8, v_1, v_3, v_5, v_7, v_0$ connected in this order. We denote this cycle by C. Then, the spatial graph $\bar{f}(K_9)$ contains C. Thus, we only need to consider the crossing changes on C.

We suffice to treat 36 cases with one crossing change of $\bar{f}(K_9)$, because C consist of nine edges.

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First, note that the isotopy type of the knot does not change after one crossing change at adjacent two edges $E(v_i, v_{i+2})$ and $E(v_{i+2}, v_{i+4})$, where all suffixes take values in $\mathbb{Z}_9$.

Next, we consider nine crossings in projection of cycle C on xy-plane. In other words, these are crossings $E(v_{i+1}, v_{i+3})$ and $E(v_{i+2}, v_{i+4})$. In this case, the spatial graph $\tilde{f}(K_9)$ contains knots with the crossing number being greater than or equal to 5 which is the cycle $v_{i+1}, v_{i+3}, v_i, v_{i+7}, v_{i+5}, v_{i+2}, v_{i+4}, v_{i+6}, v_{i+8}, v_{i+1}$ connected in this order.

Next, we consider 14 cases except the following 4 cases;

$E(v_2, v_4)$ and $E(v_1, v_8)$, $E(v_4, v_6)$ and $E(v_0, v_7)$,
$E(v_5, v_7)$ and $E(v_1, v_8)$, $E(v_1, v_3)$ and $E(v_0, v_7)$.

In these cases, the spatial graph $\tilde{f}(K_9)$ contains knots with the crossing number being greater than or equal to 5 which is the cycle $v_0, v_2, v_4, v_6, v_8, v_1, v_3, v_5, v_7, v_0$ connected in this order. Finally, we also see that spatial graph $\tilde{f}(K_9)$ contains knots with the crossing number being greater than or equal to 5 in the above four cases by connecting suitable edges.

References

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