Beyond Grötzsch's Theorem and Algorithmic applications

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A graph is *k-degenerate* if each subgraph has a vertex of degree at most k. Clearly, every k-degenerate graph is $(k+1)$-colorable. Borodin [1] was perhaps the first to suggest to improve this upper bound on the chromatic number by partitioning the vertex set into two sets which induce graphs with better degeneracy properties. Specifically he conjectured that every planar graph (which is 5-degenerate by Euler's formula) has a vertex partition whose parts induce a 1-degenerate and a 2-degenerate graph, respectively, and another a vertex partition whose parts induce a 0-degenerate and a 3-degenerate graph, respectively. These conjectures were proved in [5] and [6], respectively. In this paper we show that also Grötzsch's theorem can be proved by a degeneracy argument as follows:

Theorem 1. *Every planar graph of girth at least* 5 *can be partitioned into an independent set and a set which induces a forest.*

An old problem of Steinberg (see [2], page 43) says that every planar graph without 4-cycles and 5-cycles is 3-colorable. Examples show that it is not sufficient to exclude 4-cycles. In the next result we allow 5-cycles but we add additional conditions on the 3-cycles and the 4-cycles.

Theorem 2. *Let G be a plane graph. Assume that every triangle of G has a vertex which is on the outer face boundary and is contained in no* 4*-cycle. Assume also that the distance between any two triangles is at least* 3*. Then G has chromatic number at most* 3.

The proof just follows from the result in Theorem 1. For details, we refer the reader in [3].

Recently, we further extend the above result by proving the following.

Theorem 3. *Let G be a triangle-free graph on a surface of Euler genus g. Then G has a vertex set A with at most $1000g$ vertices such that $G-A$ is* 3*-colorable.*

Theorem 4. *Let G be a graph of girth at least* 5 *on a surface of Euler genus g. Then G has a vertex set A with at most $1000g$ vertices such that $G-A$ has a coloring*

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in of $V(G)$ in colors $1, 2$ such that the vertices of color 1 induce an independent set, and the vertices of color 2 induce a forest.

Theorems 3 and 4 will appear in [4].

The above proofs imply polynomial algorithms. In fact, the proof of Theorem 1 implies an $O(n^2)$ algorithm to partition a given planar graph of girth 5, and the proof of Theorem 4 also implies an $O(n^2)$ algorithm to give the desired conclusion, once an input graph (of girth 5 embedded into a fixed surface is given).

Furthermore, the proof of Theorem 3 implies an $O(nlogn)$ algorithm. We shall also discuss some algorithmic applications of Theorem 3.

References

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