Topology of box complexes of graphs

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In this talk, we assume that all graphs are finite, simple, undirected and connected. L. Lovász, in [4], proved Kneser's conjecture posed in [3] by using the Borsuk-Ulam Theorem. His method was further developed by J. Matoušek and G. M. Ziegler in [5], where they introduced the box complex $\mathsf{B}(G)$ of a graph $G = (V(G), E(G))$ to obtain a lower bound for the chromatic number $\chi(G)$ of G (see below).

We are interested in the relation between topology of the box complex of G and combinatorics of G. First we give a few notions. For a subset U of $V(G)$, a vertex v of G is called a *common neighbor* of U if $uv \in E(G)$ for all $u \in U$. The set of all common neighbors of U is denoted by $\mathrm{CN}_G(U)$. For subsets U_1, U_2 of $V(G)$ such that $U_1 \cap U_2 = \phi$, we define $G[U_1, U_2]$ as the bipartite subgraph of G with the bipartition $\{U_1, U_2\}$ and the edge set $\{u_1u_2 \in E(G) \,|\, u_1 \in U_1,\, u_2 \in U_2\}$. Also let $U_1 \uplus U_2 := (U_1 \times \{1\}) \cup (U_2 \times \{2\}) \subset V(G) \times \{1, 2\}$. The *box complex* of a graph G is an abstract simplicial complex with the vertex set $V(G) \times \{1, 2\}$ defined by

$$\begin{aligned}\mathsf{B}(G) = \{U_1 \uplus U_2 \,|\, U_1, U_2 \subseteq V(G),\, U_1 \cap U_2 = \phi, \\ G[U_1, U_2] \text{ is complete}, \mathrm{CN}_G(U_1) \neq \phi \neq \mathrm{CN}_G(U_2)\}.\end{aligned}$$

It admits a free simplicial $\boldsymbol{Z}_2$-action ν on $V(\mathsf{B}(G))$ defined by $u \uplus \phi \mapsto \phi \uplus u$ and $\phi \uplus u \mapsto u \uplus \phi$ (so $\mathsf{B}(G)$ is called a free $\boldsymbol{Z}_2$-complex). Hence, the $\boldsymbol{Z}_2$-*index* of $\mathsf{B}(G)$ is defined as follows:

$$\begin{aligned}\mathrm{ind}(\mathsf{B}(G)) = \min\{\, n \,|\, &\text{there exists a continuous} \\ &\text{map } f : \|\mathsf{B}(G)\| \to S^n \text{ such that } f \circ \nu = A \circ f\},\end{aligned}$$

where $S^n = \{x \in \boldsymbol{R}^{n+1} \,|\, \|x\| = 1\}$ with the antipodality A. In [5], Matoušek and Ziegler proved for any graph G, the following inequality holds:

$$\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)) + 2.$$

The difference $\chi(G) - (\mathrm{ind}(\mathsf{B}(G)) + 2)$ can be arbitrarily large.

We define a 1-dimensional abstract simplicial complex $\overline{G}$ with the vertex set $V(\mathsf{B}(G))$ as follows:

$$\overline{G} = \{\, u \uplus \phi,\, v \uplus \phi,\, \phi \uplus u,\, \phi \uplus v,\, u \uplus v,\, v \uplus u \mid uv \in E(G)\}.$$

Notice that $\overline{G}$ is a free $\boldsymbol{Z}_2$-subcomplex of $\mathsf{B}(G)$ and is a natural double covering of G. First, we present the relation between $\mathsf{B}(G)$ and $\overline{G}$ for a general graph G. It was shown, in [2], that $\mathsf{B}(G)$ is connected if and only if $\overline{G}$ is connected. It was also proved that G is bipartite if and only if $\overline{G}$ is disconnected. Then, we have the following:

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Theorem 1. *A graph G is bipartite if and only if* $\mathsf{B}(G)$ *is disconnected.*

When a graph G contains no 4-cycle, Matoušek and Ziegler, in [5], proved that there exists a $\boldsymbol{Z}_2$-retraction of $\|\mathsf{sd}\,\mathsf{B}(G)\|$ onto $\|\mathsf{sd}\,\overline{G}\|$, and in particular, $\mathrm{ind}(\mathsf{B}(G)) \leq 1$. In [1], it was shown that $\|\overline{G}\|$ is actually a strong $\boldsymbol{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$, in particular, $\mathrm{ind}(\mathsf{B}(G)) = \mathrm{ind}(\overline{G})$. Moreover, we have the following:

Theorem 2. *A graph G without 4-cycles if and only if $\|\overline{G}\|$ is a strong $\boldsymbol{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$.*

Next, for the union $G \cup H$ of two graphs G and H, we compare $\mathsf{B}(G \cup H)$ with its subcomplex $\mathsf{B}(G) \cup \mathsf{B}(H)$. One cannot hope that $\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H)$ and $\mathsf{B}(G) \cap \mathsf{B}(H) = \mathsf{B}(G \cap H)$ in general. We give a sufficient condition under which those equalities hold.

Theorem 3. *Let $G \cup H$ be the union of two graphs G and H, and assume that the intersection $G \cap H$ is of the form:*

$$V(G \cap H) = \{u_1, \cdots, u_k, v_1, \cdots, v_k\} \text{ and } E(G \cap H) = \{u_i v_i \,|\, i = 1, \cdots, k\}.$$

Further we assume that (1) $u_1, \cdots, u_k$ are endvertices of H, (2) $v_1, \cdots, v_k$ are endvertices of G and (3) the set $\{u_1, \cdots, u_k\}$ is independent in G. Then, we obtain

$$\mathsf{B}(G \cup H) = \mathsf{B}(G) \cup \mathsf{B}(H) \text{ and } \mathsf{B}(G \cap H) = \mathsf{B}(G) \cap \mathsf{B}(H).$$

For the union $G \cup H$ satisfying the condition of Theorem 3, we give an estimate of the chromatic number of $G \cup H$:

Theorem 4. *Let $G \cup H$ be the union satisfying the condition of Theorem 3.*
(1) If $k \geq 2$, we have $\chi(G \cup H) \leq \max\{\chi(G) + l_H, \chi(H)\}$, where l_H is the graph invariant defined in this talk. If $k = 1$, we have $\chi(G \cup H) = \max\{\chi(G), \chi(H)\}$.
(2) If $\max\{\mathrm{ind}(\mathsf{B}(G)), \mathrm{ind}(\mathsf{B}(H))\} \geq 1$, we have

$$\mathrm{ind}(\mathsf{B}(G \cup H)) = \max\{\mathrm{ind}(\mathsf{B}(G)), \mathrm{ind}(\mathsf{B}(H))\}.$$

If $\mathrm{ind}(\mathsf{B}(G)) = \mathrm{ind}(\mathsf{B}(H)) = 0$, we have $\mathrm{ind}(\mathsf{B}(G \cup H)) \leq 1$.

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