The minimum non-crossing edge number of graphs

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Let G be a simple and connected graph. A *drawing* of G is a figure on the plane consisting of points and curves so as to express G. We allow crossing of edges in such a drawing. In particular, a drawing of G is called a *straight drawing* of G if all edges are drawn by straight line segments. We often use the notation $d(G)$ to distinguish a given drawing of G from the abstract graph G.

In topological graph theory, there has been discussed the minimum number of crossings taken over all drawings of a graph by many people, but such a problem is difficult in general. On the other hand, if we replace the minimum with the maximum, then the problem will become easy. So we focus on those edges in a straight drawing of a graph G that contain no crossing and call the number of such edges *the non-crossing edge number* of $d(G)$ and denote it by $\mathrm{nce}(d(G))$.

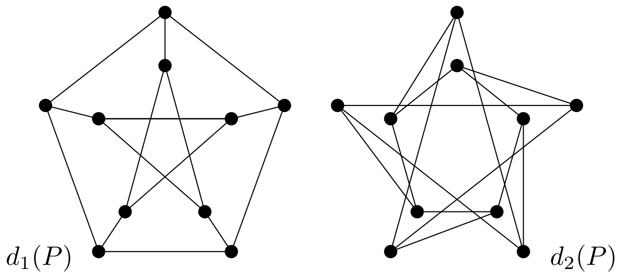

Figure 1 Straight drawings of the Petersen graph

The minimum non-crossing edge number of G is defined as the minimum value of non-crossing edge number taken over all straight drawings of G, and is denoted by $\mathrm{nce}(G)$. For example, Figure 1 presents two straight drawings of the Petersen graph P, say $d_1(P)$ and $d_2(P)$ and we have $\mathrm{nce}(d_1(P)) = 10$ and $\mathrm{nce}(d_2(P)) = 0$. This implies that $\mathrm{nce}(P) = 0$.

Theorem 1. *The minimum non-crossing edge number of the complete graph* K_n

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with n vertices can be given as follows:

$$\mathrm{nce}\,(K_n) = \begin{cases} 0 & (n = 1) \\ 1 & (n = 2) \\ 3 & (n = 3) \\ 4 & (n = 4) \\ 5 & (n \geq 5, \neq 7) \\ 6 & (n = 7) \end{cases}$$

Generalizing our arguments to prove this formula, we can give an upper bound for $\mathrm{nce}(G)$ of a graph G by its chromatic number $\chi(G)$.

Theorem 2. *Let G be a graph, isomorphic to neither $K_{1,n}$ nor K_7. Then we have:*

$$\mathrm{nce}(G) \leq \min\{\chi(G), 5\}$$

The equality in the above inequality is attained when G is a complete $\chi(G)$-partite graph.